Journal of Mathematics and Physics

(Founded by C. L. E. Moore)

Volume XXVIII

1949 - 1950

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ON BACKFLOW OF A VISCOUS FLUID IN A DIVERGING CHANNEL*

By Milton Abramowitz

1. Introduction. Viscous flow between parallel plane walls may be described by the Poiseuille flow distribution $u = U(1-\eta^2/a^2)$ when the Reynolds number does not exceed a certain limit and the motion remains steady. Slight convergence of the walls tends to stabilize the laminar flow and a complete description of the flow has been given on the basis of the boundary layer theory [1]. However, if the walls of the channel diverge the situation is reversed. Due to the divergence there is a decrease in velocity. The consequent increase in pressure is superposed on the frictional pressure in such a manner that the direction of flow will reverse itself when the resultant pressure gradient becomes negative. The problem of two dimensional steady flow has been studied under the simplifying assumption that the entrant flow is radial and the boundary of the channel is rectilinear [2]. The resulting solution is expressed in terms of elliptic functions. However, the practical value of this solution is limited since the fluid will not generally enter the channel in radial fashion. Its chief claim to importance is the fact that it is one of the few instances where the Navier-Stokes equations in polar coordinates can be solved exactly. Another method of attack has been given by Blasius [3]. Under non-radial initial conditions and employing a method of approximations he demonstrates a symmetrical solution of the two-dimensional Navier-Stokes equations in Cartesian coordinates. Blasius found that backflow will occur in a divergent channel when $mR = 105/8$ where m = slope of the channel and R is the Reynolds number. We shall first justify the result obtained by Blasius. Secondly, employing his procedure we shall carry the approximation another step to refine the critical condition for backflow. Finally, using a different approach we shall determine an improved solution. In particular we shall be able to calculate the point of detachment from the backflow criterion.

2. Equation of Flow. Consider a diverging channel whose boundary is given by the relation $\eta = \pm\varphi(\xi)$, $\varphi(0) = \pm a$.

We assume that the fluid enters the divergent section of the channel at $\xi = 0$ after having passed through a straight portion in which the Poiseuille flow has developed. The Navier-Stokes equations for the flow are

$$\psi_\eta\omega_\xi - \psi_\xi\omega_\eta = \nu\nabla^2\omega, \qquad \omega = \nabla^2\psi \tag{2.1}$$

where

$$u = \psi_\eta\,; \qquad v = -\psi_\xi \tag{2.2}$$

* The author wishes to express his gratitude to Professor K. O. Friedrichs for his advice in the preparation of this paper. The content of this paper was submitted as partial fulfillment of the requirements for the degree of doctor of philosophy at New York University.

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and the boundary conditions are

$$\psi = aU\left(\frac{\eta}{a} - \frac{1}{3}\frac{\eta^3}{a^3}\right); \qquad \psi_\xi = 0 \text{ for } \xi = 0$$

(2.3)

$$\psi = \tfrac{2}{3}aU; \qquad \psi_\eta = 0 \quad \text{for} \quad \eta = \pm\varphi(\xi).$$

The boundary condition at $\xi = 0$ expresses the fact that the fluid entering the divergent portion of the channel is characterized by the Poiseuille flow. The condition along $\eta = \pm\varphi(\xi)$ requires that the boundary shall be a streamline for the flow. In addition we impose the condition that $\psi(\xi, \eta)$ shall be finite for $\xi = \infty$.

3. Transformation of Equations. We note that the boundary conditions (2.3) refer to the curvilinear boundary $\eta = \pm\varphi(\xi)$ of the divergent channel. In order to work in the same fixed domain for various types of channels we set

$$\delta\xi = x; \qquad \eta = y\varphi(\xi) = y\varphi(\delta^{-1}x), \qquad \varphi(0) = \pm a. \tag{3.1}$$

The values of the parameter δ is not fixed as yet. At a later stage we shall be able to attach proper significance to it. However, if we particularize the types of channels to be considered to the form $\eta = \varphi_\delta(\xi) = \varphi(\delta\xi)$ then we obtain $\varphi_\delta(\xi) = \varphi(\delta\xi) = \varphi(x)$ a function which is independent of δ so that $\varphi_\xi = \delta\varphi_x$. The differential equation (2.1) will then become

$$\begin{aligned}\delta\varphi\{\omega_x\psi_y - \omega_y\psi_x\} = \nu\{&\delta^2\varphi^2\omega_{xx} + (1 + \delta^2y^2\varphi_x^2)\omega_{yy} \\ &+ y\delta^2(2\varphi_x^2 - \varphi\varphi_{xx})\omega_y - 2\delta^2\varphi\varphi_x\omega_{xy}\}\end{aligned} \tag{3.2}$$

$$\varphi^2\omega = \delta^2\varphi^2\psi_{xx} + (1 + \delta^2y^2\varphi_x^2)\psi_{yy} + y\delta^2(2\varphi_x^2 - \varphi\varphi_{xx})\psi_y - 2\delta^2\varphi\varphi_x\psi_{xy}$$

and the boundary conditions (2) become

$$\psi = aU(y - \tfrac{1}{3}y^3), \tag{3.3}$$

$$\psi_x = aU\varphi_x(0)(y - y^3) \text{ for } x = 0, \tag{3.4}$$

$$\psi = \tfrac{2}{3}aU, \tag{3.5}$$

$$\psi_y = 0 \text{ for } y = \pm 1. \tag{3.6}$$

4. Condition for Backflow. In order that the point $\xi = \xi_0$ shall be a point of detachment there must be a streamline $\eta = H(\xi)$ such that $H(\xi_0) = \varphi(\xi_0)$ and $H_\xi(\xi_0) \neq \varphi_\xi(\xi_0)$. These two conditions imply that the boundary $\eta = \varphi(\xi)$ and the streamline $\eta = H(\xi)$ intersect at $\xi = \xi_0$. The second relation implies that the boundary $\eta = \varphi(\xi)$ is a streamline which separates into two distinct parts at $\xi = \xi_0$. Since $H(\xi)$ and $\varphi(\xi)$ are streamlines

$$\psi(\xi, H(\xi)) = \psi(\xi, \varphi(\xi)) = \psi(\xi_0, \varphi(\xi_0)) = \text{constant} \tag{4.1}$$

$$\psi_\xi + \psi_\eta H_\xi(\xi) = 0, \qquad \psi_{\xi\xi} + 2\psi_{\xi\eta}H_\xi + \psi_\eta H_{\xi\xi} + \psi_{\eta\eta}H_\xi^2 = 0. \tag{4.2}$$

When $\xi = \xi_0$ we have $H(\xi_0) = \varphi(\xi_0)$ and $\psi_\eta = 0$ from the boundary conditions, so that

$$\psi_{\xi\xi} + 2\psi_{\xi\eta}\varphi_\xi + \psi_{\eta\eta}H_\xi^2 = 0. \tag{4.3}$$

Furthermore on the boundary we have $\psi_\xi = 0, \psi_\eta = 0$. Differentiating with respect to ξ we get

$$\psi_{\xi\xi} + \varphi_\xi \psi_{\xi\eta} = 0; \qquad \psi_{\xi\eta} + \varphi_\xi \psi_{\eta\eta} = 0. \tag{4.4}$$

Eliminating $\varphi_{\xi\xi}$ and $\psi_{\xi\eta}$ from the last three equations we get

$$\psi_{\eta\eta}\{\varphi_\xi(\xi_0) - H_\xi(\xi_0)\}^2 = 0. \tag{4.5}$$

By hypothesis, the expression in brackets is different from zero so that the backflow becomes $\psi_{\eta\eta} = 0$. It is evident that under the transformation in (3.1) the condition in the (xy) plane is $\psi_{yy} = 0$.

5. Blasius Method. On the basis of the preliminaries we are now able to obtain a justification of Blasius' solution. Let $\eta = \pm\varphi(\xi)$ with $\varphi(0) = \pm a$ represent a particular channel. We can relate this channel to the straight channel $\eta = \pm a$ by a continuous set of channels $\eta = \varphi_m(\xi) = \varphi(m\xi)$. As $m \to 0$ these channels clearly converge to the straight channel $\varphi(\xi) = a$. We identify the arbitrary parameter previously referred to as δ by setting $\delta = m$ and therefore the function $\varphi_m(\xi) = \varphi(m\xi) = \varphi(m\delta^{-1}x) = \varphi(x)$ when considered as a function of x is independent of m together with its derivatives with respect to x. Blasius assumed the boundary of the channel was of the form $\varphi_m(\xi) = ae^{m\xi}$. However, if m is small enough $e^{m\xi} \sim 1 + m\xi$ and the solution for the linear case will be sufficient to characterize the flow for the boundary of small exponential divergence. In order to determine the solution of (3.2) we assume the following expansions in terms of the parameter m.

$$\begin{aligned} \psi &= \psi^0 + m\psi^1 + m^2\psi^2 + \cdots \\ \omega &= \omega^0 + m\omega^1 + m^2\omega^2 + \cdots . \end{aligned} \tag{5.1}$$

Thus substituting (5.1) into (3.2) and neglecting terms of order m^3 or higher we obtain the following systems of equations by equating powers of m.
System I:

$$\nu\omega^0_{yy} = 0; \qquad \varphi^2\omega^0 = \psi^0_{yy} \tag{5.2}$$

with the boundary conditions

$$\psi^0 = \tfrac{3}{4}aU; \qquad \psi^2_y = 0 \text{ for } y = \pm 1 \tag{5.3}$$

The solution of this system is given by the Poiseuille flow

$$\psi^0 = \tfrac{3}{4}aU(y - \tfrac{1}{3}y^3); \qquad \omega^0 = -(2aU/\varphi^2)y. \tag{5.4}$$

When m is small backflow will not occur near the mouth of the channel since divergence from the rectilinear channel will be small. Thus we are interested in the behavior of the flow away from the entrant portion. This has been the motivation for the transformation $m\xi = x$. A fixed section $0 \le \xi \le \xi_0$ goes over by the transformation into a varying section $0 \le x \le m\xi$ and in the limit shrinks to a point $x = 0$. Therefore we cannot expect that the limit function for $m \to 0$ shall satisfy the boundary conditions at $x = 0$ which the solution for $m > 0$

satisfies at $x = 0$. This is consistent with the fact that the solution of the limit differential equation is already determined without imposing a boundary condition at $x = 0$. This situation is similar to that occurring in the boundary layer theory where a boundary condition is lost in the limiting process. The expression given in (5.4) above has been so obtained and the solution obtained is valid for values of x away from the entrant portion.

System II:

$$\nu\omega^1_{yy} = \varphi\{\omega^0_x\psi^0_y - \omega^0_y\psi^0_x\}, \qquad \varphi^2\omega^1 = \psi^1_{yy} \tag{5.5}$$

with the boundary conditions

$$\psi^1 = 0; \qquad \psi^1_y = 0 \text{ for } y = \pm 1. \tag{5.6}$$

The solution of this system is

$$\begin{aligned} \psi^1(x, y) &= \frac{a^2U^2\varphi_x}{\nu}\left\{\tfrac{1}{42}y - \tfrac{1}{210}y^3 + \tfrac{1}{360}y^5 - \tfrac{1}{2160}y^7\right\} \\ \omega^1(x, y) &= \frac{a^2U^2\varphi_x}{\nu\varphi^2}\left\{-\tfrac{1}{8}\tfrac{1}{5}y + \tfrac{2}{3}y^3 - \tfrac{1}{5}y^5\right\}. \end{aligned} \tag{5.7}$$

System III:

$$\begin{aligned} \varphi\{\omega^1_x\psi^0_y + \omega^0_x\psi^1_y - \psi^0_x\omega^1_y - \psi^1_x\omega^0_y\} &= \nu\{\omega^2_{yy} + \varphi^2\omega^0_{xx} + y^2\varphi^2_x\omega^0_{yy} \\ &\quad + y(2\varphi^2_x - \varphi\varphi_{xx})\omega^0_y - 2y\varphi\varphi_x\omega^0_{xy}\} \\ \varphi^2\omega^2 = \varphi^2\psi^0_{xx} + y^2\varphi^2_x\psi^0_{yy} + \psi^2_{yy} &+ y(2\varphi^2_x - \varphi\varphi_{xx})\psi^0_y - 2y\varphi\varphi_x\psi^0_{xy} \end{aligned} \tag{5.8}$$

with the boundary conditions

$$\psi^2 = 0; \qquad \psi^2_y = 0 \text{ for } y = \pm 1. \tag{5.9}$$

Up to this point the special nature of the boundary of the channel has not been used. However, in order to carry out the integration of System III we assume that the boundary of the channel is a straight line so that $\varphi(x) = a + x$, $\varphi_m(\xi) = a + m\xi$. In this case the terms involving φ_{xx} drop out of the equations of System III and the solution will be independent of x. Furthermore, in order to insure the occurrence of backflow, we assume that the Reynolds number $R = \dfrac{Ua}{\nu}$ characterizing the flow is large so that terms involving R^{-1} may be neglected. If we substitute the known functions in (5.8) we get for the particular rectilinear boundary

$$\frac{\partial^4\psi^2}{\partial y^4} = \frac{a^3U^3}{\nu^2}\left\{\left(\frac{76}{105} + \frac{48}{R^2}\right)y - \frac{272}{105}y^3 + \frac{12}{5}y^5 - \frac{8}{15}y^7\right\}. \tag{5.10}$$

By assumption the term $48/R^2$ is small, so that it may be neglected. Integration yields

$$\begin{aligned} \psi^2(x, y) = \frac{a^3U^3}{100\nu^2}\{&.19755378y - .56496943y^3 + .60317460y^5 \\ &- .30839002\,y^7 + .07936508\,y^9 - .00673401\,y^{11}\} \end{aligned} \tag{5.11}$$

$$\psi^2_{yy}(x, 1) = .0069484(a^3U^3/\nu^2).$$

The stream function $\psi(x, y)$ is then obtained from the expression

$$\psi(x, y) = aU\{\psi^0 + mR\psi^1 + (mR)^2\psi^2\}. \tag{5.12}$$

6. Discussion of Backflow. With the results obtained we are now able to determine the condition for backflow. In the xy-plane, the condition for backflow is $\psi_{yy}(x, y) = 0$ for $y = \pm 1$. The expression for $\psi(x, y)$ is given in (5.1) so that when $y = \pm 1$ we should determine a value of m for each particular value of x. The condition for backflow is

$$\psi_{yy}(x, y) = \psi^0_{yy}(x, y) + m\psi^1_{yy}(x, y) + m^2\psi^2_{yy}(x, y) + \cdots = 0 \text{ for } y = \pm 1. \tag{6.1}$$

Since we have carried the expansion for $\psi(x, y)$ up to terms of order m^2, (6.1) will be a quadratic equation in mR. The parameter m is the slope of the boundary of the channel $\eta = \varphi_m(\xi) = a + m\xi$. After substituting the values of ψ^0_{yy}, ψ^1_{yy}, ψ^2_{yy} in (6.1) we find that $\psi_{yy}(x, y)$ is independent of x for all these channels, so that the position of the point of detachment cannot be fixed. However, our later investigation will show that it actually takes place at a definite point $x = x_s$ which can be determined from the new relations. The explicit expression for (6.1) is

$$1 - .076190\,(mR) - .003474\,(mR)^2 = 0 \tag{6.2}$$

The positive root of this equation is $mR = 9.24$. Blasius' solution included only the first two terms of (5.1) and he found the critical value $mR = 13.125$.

In an experimental investigation carried out by Patterson it was found that the value of mR obtained by Blasius was too large. Thus, the critical value obtained here will be more accurate but not very accurate anyway. The approximation can be determined to any desired degree of accuracy by merely evaluating further terms in the expansion for $\psi(x, y)$. Furthermore, since Blasius stated in his discussion that his result was to be valid for channels of small exponential divergence, Patterson went to the trouble of constructing his channel so that it would conform with the Blasius theory. Actually as we have shown the same experimental results could have been obtained if a diverging channel with rectilinear boundaries were studied.

If we criticize the Blasius results from the view-point of adequacy, the first objection to be offered is that his critical value for mR is much too large. Furthermore, since the condition he obtained was independent of x, it gives no clue to the position at which the backflow occurs. Actually, as our later investigation will show, his result corresponds to the situation where backflow occurs at $\xi = \infty$.

7. Formulation of New Method. We will now try to develop a theory in which the backflow is found to occur at some finite point. To this end consider the parameter involved in the problem. First of all the Reynolds number R must be large for practical purposes. Secondly the parameter m which is the slope of the rectilinear boundary should be allowed to take on values which are not necessarily small. In the discussion of the Blasius method it was assumed that the Reynolds number was fixed at first and then determined the particular

channel from the set of channels $\varphi_m(\xi) = a + m\xi$. Thus, if $S(R^{-1}, m; \xi, \eta)$ represents the solution, it can be written in the form

$$S(R^{-1}, m; \xi, \eta) = S(R^{-1}, 0; \xi, \eta) + mS_m(R^{-1}, 0; \xi, \eta) + \frac{m^2}{2!} S_{mm}(R^{-1}, 0; \xi, \eta) + \cdots .$$

From the condition obtained for backflow the parameters R and m are dependent on one another but independent of the point of detachment. Examining the Blasius solution we note that the boundary of the diverging channels were referred to the fixed straight channel by means of the transformation,

$$m\xi = x; \qquad \eta = y\varphi(\xi); \qquad \varphi(0) = \pm a.$$

The diverging channels were assumed to be of the form $\varphi_m = \varphi(m\xi) = a + m\xi$ where m is the slope of the channel. Then for fixed Reynolds number R an expansion for ψ was taken as

$$\psi = \psi^0 + m\psi^1 + m^2\psi^2 + \cdots .$$

The slope m was taken to be small so that the above expression would give a satisfactory representation up to terms of order m^2. The solution was then finally obtained for values of R^{-1} that are small.

In order to obtain an improvement over the Blasius solution, we shall refer the boundary of the diverging channels to the fixed boundary of the straight channel by first choosing $\delta = (aR)^{-1}$ so that the transformation shall be

$$\xi = aRx; \qquad \eta = y\varphi(\xi) = y\phi(aRx), \qquad \phi(0) = \pm a.$$

We assume that the set of channels $\eta = \phi_s(\xi)$ are of the form $\eta = a\varphi(\xi/aR) = a\varphi(x)$. Then, if $\varphi(x) = 1 + \epsilon\varphi_1(x)$ it is possible to determine the parameter ϵ as follows: The slope of the channel is $m = d\eta/d\xi$. Thus, for $\xi = 0$ we get

$$m = \frac{d\eta}{d\xi} = \frac{a\varphi'(0)}{aR} = \frac{\varphi'(0)}{R}.$$

Therefore $mR = \varphi'(0)$ is fixed and if $\varphi = 1 + \epsilon\varphi_1(x)$, we have $\epsilon = mR/\varphi_1'(0)$.

Since we are interested in flows for which R^{-1} is small and have chosen $\delta = (aR)^{-1}$, we first let $\delta \to 0$ in (3.2) and obtain

(7.1) $\varphi\{\omega_x\psi_y - \omega_y\psi_x\} = U\omega_{yy}$, $\varphi^2\omega = \psi_{yy}$; (7.2) $\psi = aU(y - \frac{1}{3}y^3)$ for $x = 0$

(7.3) $\psi_x = aU\varphi_x(0)(y - y^3)$ for $x = 0$; (7.4) $\psi = \frac{2}{3}aU$ for $y = \pm 1$

(7.5) $\psi_y = 0$ for $y = \pm 1$.

The solution of the system of differential equations (7.1) to (7.4) is the problem that will occupy us in this paper from now on. However, as in the Blasius case a fixed section $0 \leq \xi \leq \xi_0$ goes into $0 \leq x \leq \xi_0/aR$. When $R \to \infty$ the fixed section goes into $x = 0$ so that we cannot hope to satisfy all the conditions at $x = 0$. For this reason we shall discard (7.3).

In order to determine the solution of (7.1)–(7.4) we make use of a method of approximations. We choose as the parameter the quantity defined above and set

$$(7.6) \qquad \psi = \psi^0 + \epsilon\psi^1 + \epsilon^2\psi^2 + \cdots \qquad \omega = \omega^0 + \epsilon\omega^1 + \epsilon^2\omega^2 + \cdots.$$

Substituting in (7.1) we get the following systems of equations and their corresponding boundary conditions.

System I:

$$(7.7)\ \omega^0_x\psi^0_y - \omega^0_y\psi^0_x = U\omega^0_{yy}; \qquad \omega^0 = \psi^0_{yy}; \qquad (7.8)\ \psi^0 = aU(y - \tfrac{1}{3}y^3) \text{ for } x = 0$$

$$(7.9)\ \psi^0 = \tfrac{2}{3}aU \text{ for } y = \pm 1; \qquad (7.10)\ \psi^0_y = 0 \text{ for } y = \pm 1$$

System II:

$$(7.11)\ \omega^0_x\psi^1_y + \omega^1_x\psi^0_y - \omega^0_y\psi^1_x - \omega^1_y\psi^0_x = U\omega^1_{yy} \qquad \omega^1 = \psi^1_{yy} - 2a\varphi_1(x)$$

$$(7.12)\ \psi^1 = 0 \text{ for } x = 0; \qquad (7.13)\ \psi^1 = 0 \text{ for } y = \pm 1$$

$$(7.14)\ \psi^1_y = 0 \text{ for } y = \pm 1$$

System III:

$$(7.15)\ \omega^0_x\psi^2_y + \psi^0_y\omega^2_x + \omega^1_x\psi^1_y - \omega^0_y\psi^2_x - \psi^0_x\omega^2_y - \omega^1_y\psi^1_x + aU\varphi_1\omega^1_{yy} = U\omega^2_{yy}$$

$$\omega^2 = \psi^2_{yy} - 2a\varphi_1\omega^1 - a^2\varphi_1\omega^0$$

$$(7.16)\ \psi^2 = 0 \text{ for } x = 0; \qquad (7.17)\ \psi^2 = 0 \text{ for } y = \pm 1$$

$$(7.18)\ \psi^2_y = 0 \text{ for } y = \pm 1.$$

8. Solution of Systems I and II. The solution of System I is $\psi^0 = aU(y - \frac{1}{3}y^3)$ agreeing with that obtained by Blasius for his first approximation.

If in order to determine the solution of system II, we substitute in (7.11) the explicit expressions for ψ^0 and ω^0 we obtain

$$(8.1) \qquad \omega^1_{yy} - (1 - y^2)\omega^1_x - 2\psi^1_x = 0; \qquad \omega^1 = \psi^1_{yy} + 4aU\varphi_1(x)y.$$

In order to determine a solution of this system satisfying the boundary conditions (8.1) we first note that the function $\psi^1(x; y) = aU\varphi_1(x)(y - y^3)$ satisfies the differential equation (8.1), boundary conditions (7.11), (7.14) but not (7.13). Therefore putting

$$(8.2) \qquad \psi^1(x, y) = f(x, y) + aU\varphi_1(x)(y - y^3)$$

and eliminating ω^1 from (8.1) we obtain

$$(8.3) \qquad \frac{\partial^4 f}{\partial y^4} - (1 - y^2)\frac{\partial^3 f}{\partial x^2 \partial y} - 2\frac{\partial f}{\partial x} = 0$$

and

$$(8.4)\ f = 0, \text{ for } x = 0; \qquad f = 0, \quad \frac{\partial f}{\partial y} = 2aU\varphi_1(x) \text{ for } y = \pm 1.$$

9. Solution of (8.3) as a contour integral. In order to obtain a solution of (8.3) and (8.4) we assume that $f(x, y)$ may be expressed as a contour integral, the path of integration being an appropriate line parallel to the imaginary axis. Thus

$$f(x, y) = \frac{1}{2\pi i}\int_{\uparrow} e^{\lambda x} H(\lambda) G(y, \lambda)\, d\lambda \tag{9.1}$$

where $G(y, \lambda)$ is such that $G_y(y, \lambda) = 1$ for $y = \pm 1$. Then

$$f_y(x, 1) = \frac{1}{2\pi i}\int_{\uparrow} e^{\lambda x} H(\lambda)\, d\lambda. \tag{9.2}$$

Furthermore, since $f_y(x, 1) = 2aU\varphi_1(x)$ is given, the function $H(\lambda)$ can be determined by the Laplace transformation as

$$H(\lambda) = \int_0^\infty e^{-x} f_y(x, 1)\, dx = 2aU\int_0^\infty e^{-x\lambda}\varphi_1(x)\, dx. \tag{9.3}$$

Substituting the integral for $f(x, y)$ into (8.3) we see that all the conditions are satisfied if

(9.4) $G^{iv} = \lambda\{(1 - y^2)G'' + 2G\}$ and $G(y, \lambda)$ satisfies the conditions

(9.5) $G(0) = G''(0) = 0$; (9.6) $G(1) = 1$; (9.7) $G'(1) = 1$.

The condition (9.5) is a consequence of the fact that the flow shall be symmetrical about the center line, $y = 0$ of the channel, (9.6) results from the condition $f(x, \pm 1) = 0$ and (9.7) from the definition of $G(y, \lambda)$ in (9.1).

In order to determine $G(y, \lambda)$ it is convenient to set

$$G(y, \lambda) = g(y, \lambda)/g'(1, \lambda)$$

where $g(y, \lambda)$ satisfies (9.4), (9.5), (9.6) but not (9.7). Instead we require that[1]

$$g'(0, \lambda) = 1. \tag{9.8}$$

The integral in (9.1) may be evaluated by the theorem of residues once we know the poles of the integrand. The original path of integration along the imaginary axis is deformed by shifting it to $\lambda = -\infty$. For the sake of simplicity we restrict the function $\varphi_1(x)$ so that singularities due to $H(\lambda)$ will be poles. Thus, in the shifting process if we take account of the singularities we will be able to obtain an explicit expression for (9.1). If $\lambda = \lambda_i$ are the zeros of $g'(1, \lambda)$ and $\lambda = a_i$ are the poles of $H(\lambda)$ we get

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi i}\int_{\uparrow} e^{x\lambda} H(\lambda)\,\frac{g(y, \lambda)}{g'(1, \lambda)}\, d\lambda \\ &= \sum_{n=1}^{\infty} e^{x\lambda_n} H(\lambda_n)\,\frac{g(y, \lambda_n)}{g_\lambda'(1, \lambda_n)} + \sum_i \operatorname{Res}\,\frac{e^{xa_i} H(a_i) g(y, a_i)}{g_\lambda'(1, a_i)}. \end{aligned} \tag{9.9}$$

Actually, the poles $\lambda = \lambda_i$ will be seen to be situated on the negative real axis and they must be determined as the eigenvalues together with their corresponding eigen-functions as solutions of the system

$$g^{iv} = \lambda\{(1 - y^2)g'' + 2g\} \tag{9.10}$$

[1] We write $\partial g(y, \lambda)/\partial y = g^1(y, \lambda)$ and $\partial g(y, \lambda)/\partial\lambda = g_\lambda(y, \lambda)$.

(9.11) $g(0) = g''(0) = 0$; (9.12) $g(1) = 0, g'(1) = 0$; (9.13) $g'(0) = 1$.

10. Determination of the Eigenvalues. In order to determine the successive eigenvalues we shall employ iteration methods.

To find the first eigenvalues we assume the following expansion for the first eigen function,

$$g_0(y, \lambda_1) = y - 2y^3 + y^5 \tag{10.1}$$

This function satisfies (9.11) and (9.12) but not the differential equation (9.10). Substituting in the right member of (9.10) we get

$$g_1^{iv}(y, \lambda_1) = \lambda_1\{-10y + 28y^3 - 18y^5\}. \tag{10.2}$$

Integrating and determining the constants of integration so that (9.11) and (9.12) are satisfied we obtain

$$g_1(y, \lambda_1) = \lambda_1\left\{\frac{29}{840}y - \frac{19}{210}y^3 + \frac{1}{12}y^5 - \frac{1}{30}y^7 + \frac{1}{168}y^9\right\}. \tag{10.3}$$

We are concerned with determining the eigen function $g(y, \lambda_1)$ corresponding to the fixed eigenvalue λ_1. Therefore in order to completely determine $g(y, \lambda_1)$ we make use of (9.13) namely, $g'(0, \lambda) = 1$ for all λ. This condition affords us a criterion to fix the value of λ_1 at each step of the iteration process. The functions $g_n(y, \lambda_1)$ which we are obtaining in the iteration process are approximations to the eigen function. Furthermore, if $g_n(y, \lambda_1)$ is an approximation to the eigenfunction $g(y, \lambda_1)$, the value obtained for λ_1 by satisfying (9.13) must be an approximation to the eigenvalue. Returning to the approximation $g_1(y, \lambda_1)$ we find $\frac{d}{dy}g_1(0, \lambda_1) = \frac{-29\lambda_1}{840} = 1$ so that $\lambda_1 = -\frac{840}{29} = -28.9$. We then obtain

$$g_1(y, \lambda_1) = y - \frac{76}{29}y^3 + \frac{70}{29}y^5 - \frac{140}{29}y^7 + \frac{5}{29}y^9 \tag{10.4}$$

Continuing the process until the approximation to λ_1, and $g(y, \lambda_1)$ remain unchanged (to the order of accuracy indicated) we obtain

$$\begin{aligned}\lambda_1 = -28.2210;\quad g(y, \lambda_1) = y &- 3.01266y^3 + 3.78064y^5 - 2.94517y^7 \\ &+ 1.78945y^9 - .87886y^{11} + .36498y^{13} - .13081y^{15} \\ &+ .04134y^{17} - .01166y^{19} + .00297y^{21} - .00069y^{23} + .00014y^{25}.\end{aligned} \tag{10.5}$$

In order to evaluate the integral in (9.9) by the residue theorem it is necessary to evaluate $(d/dy)g_\lambda(y, \lambda)$ for $y = 1$ at each of the poles. Differentiating (9.10), (9.11) and (9.12) with respect to λ we get

$$d^4g_\lambda/dy^4 = \lambda\{(1 - y^2)d^2g_\lambda/dy^2 + 2g_\lambda\} + \{(1 - y^2)d^2g/dy^2 + 2g\} \tag{10.6}$$

$$g_\lambda(0) = (d/dy)g_\lambda(0) = (d^2/dy^2)g_\lambda(0) = g_\lambda(1) = 0 \tag{10.7}$$

where $g(y, \lambda)$ is the solution of (9.10), (9.11) and (9.12). Since the values of λ are considered as given in (10.6) and (10.7) the function $g_\lambda(y, \lambda)$ can therefore be

determined for each of the successive eigenvalues. The actual solution of (10.6) is obtained by iteration in a method similar to the one described previously. Finally, by differentiating with respect to y and setting $y = 1$ we will obtain $(d/dy)g_\lambda(1, \lambda)$. Carrying out the solution for $\lambda = \lambda_1 = -28.2210$ we get

$$\begin{aligned}(10.8)\quad g_\lambda(y, \lambda_1) = {} & .15240y^3 - .34901y^5 + .35935y^7 - .26288y^9 + .14980y^{11} \\ & - .07029y^{13} + .02801y^{15} - .00971y^{17} + .00298y^{19} \\ & - .00082y^{21} + .00020y^{23} - .00004y^{25}\end{aligned}$$

from which we find $(d/dy)g_\lambda(1, \lambda) = -.1059$.

In order to determine the further eigenvalues $\lambda_n (n > 1)$ and eigenfunctions $g(y, \lambda)$ consider the differential equation

$$(10.9)\quad (d^4/dy^4)g(y, \lambda_m) = \lambda_m\{(1 - y^2)(d^2/dy^2)g(y, \lambda_m) + 2g(y, \lambda_m)\} = \lambda_m Lg(\lambda_m)$$

$$(10.10)\quad (d^4/dy^4)g(y, \lambda_n) = \lambda_n\{(1 - y^2)(d^2/dy^2)g(y, \lambda_n) + 2g(y, \lambda_n)\} = \lambda_n Lg(\lambda_n).$$

The adjoint differential equations are

$$(10.11)\quad (d^4/dy^4)\bar{g}(y, \lambda_m) = \lambda_m[\{(1 - y^2)\bar{g}(y, \lambda_m)\}'' + 2\bar{g}(y, \lambda_m)] = \lambda_m \bar{L}\bar{g}(\lambda_m)$$

$$(10.12)\quad (d^4/dy^4)\bar{g}(y, \lambda_n) = \lambda_n[\{(1 - y^2)\bar{g}(y, \lambda_n)\}'' + 2\bar{g}(y, \lambda_n)] = \lambda_n \bar{L}\bar{g}(\lambda_n).$$

The boundary condition for the adjoint equations are the same as in (9.11) and (9.12). Multiplying (10.9) by $\bar{g}(y, \lambda_n)$ and (10.10) by $g(y, \lambda_m)$ subtracting and integrating, we obtain

$$\begin{aligned}(10.13)\quad & \int_0^1 \{\bar{g}(\lambda_n)g^{iv}(\lambda_m) - g(\lambda_m)\bar{g}^{iv}(\lambda_n)\}\, dy \\ & = \lambda_m \int_0^1 \bar{g}(\lambda_n)Lg(\lambda_m)\, dy - \lambda_n \int_0^1 g(\lambda_m)\bar{L}\bar{g}(\lambda_n)\, dy.\end{aligned}$$

Integrating by parts we find that the left member of (10.13) vanishes so that we have

$$(10.14)\quad \lambda_m \int_0^1 \bar{g}(\lambda_n)Lg(\lambda_m)\, dy = \lambda_n \int_0^1 g(\lambda_n)\bar{L}\bar{g}(\lambda_n)\, dy.$$

Integration by parts shown that the two integrals in (10.14) are equal. Since $\lambda_m \neq \lambda_n$ the equality can hold only if both integrals vanish. Thus, we have

$$(10.15)\quad \int_0^1 g(\lambda_m)\bar{L}\bar{g}(\lambda_n)\, dy = 0 \quad \text{if} \quad m \neq n.$$

This relation provides us with an "orthogonality" condition with the aid of which the successive eigenfunctions can be determined. Suppose $\lambda_1, \lambda_2, \cdots \lambda_{n-1}$ and $g(\lambda_1), \cdots g(\lambda_{n-1})$ are known, having been determined by any means. Also let $\lambda_n^{(k)}$ and $g_n^{(k)}(\lambda_n^{(k)})$ be an approximation to λ_n and $g(\lambda_n)$ satisfying the boundary conditions (9.11) and (9.12). Then $g^{(k+1)}(\lambda_n^{(k+1)})$ can be found in the following manner. We assume that $g^{(k)}(\lambda_n^{(k)})$ has been obtained from the differential equation as the integral

$$g^{(k)}(y, \lambda_n^{(k)}) = \lambda_n^{(k)}\left\{ay + by^3 + \int\int\int\int Lg^{(k-1)}(y, \lambda_n^{(k-1)})\, dy\, dy\, dy\, dy\right\}$$

where a and b have been determined so that $g^{(k)}(y, \lambda_n^{(k)})$ satisfies the boundary conditions. Then we write

$$g^{(k+1)}(y, \lambda_n^{(k+1)}) = \alpha_1^{(k)} g(\lambda_1) + \cdots + \alpha_{n-1}^{(k)} g(\lambda_{n-1}) + g^{(k)}(\lambda_n^{(k)})$$

where $\alpha_1^{(k)}, \cdots, \alpha_{n-1}^{(k)}$ are constants which can be determined using (10.15). We multiply by $\bar{L}\bar{g}(\lambda_m)$ and integrate from 0 to 1. Then by (10.15) the integrals in which $m \neq n$ will vanish and we get

$$\int_0^1 g^{(k+1)}(y, \lambda_n^{(k+1)}) \bar{L}\bar{g}(\lambda_m)\, dy = \alpha_m^{(k)} \int_0^1 g(y, \lambda_m) \bar{L}\bar{g}(\lambda_m)\, dy + \int_0^1 g^{(k)}(y, \lambda_n^{(k)}) \bar{L}\bar{g}(y, \lambda_m)\, dy.$$

If the approximation method is convergent the left member of this relation will vanish as $k \to \infty$. We therefore assume that the left member vanishes at each step in the process and obtain an approximation to. Accordingly we set

$$\alpha_m^{(k)} = \frac{\int_0^1 g^{(k)}(y, \lambda_m^{(k)}) \bar{L}\bar{g}(y, \lambda_m)\, dy}{\int_0^1 g(y, \lambda_m) \bar{L}\bar{g}(y, \lambda_m)\, dy}.$$

It is clear that if $g^{(k)}(y, \lambda_n^{(k)}) \to g(y, \lambda_n)$ that $\alpha_m^{(k)} \to 0$. The procedure just outlined is complete except for the determination of the eigenvalue approximations $\lambda_n^{(k)}$. To achieve this we will employ a different method than the one used for the determination of λ_1. The procedure will be explained later.

In order to illustrate the foregoing we shall determine λ_2 and $g(y, \lambda_2)$ since they are necessary for our development. Consequently it will be necessary to determine $\bar{g}(y, \lambda_1)$ and $\bar{L}\bar{g}(y, \lambda_1)$. This is accomplished by solving (10.11) for $n = 1$. The actual solution is found by iteration in a similar fashion to that employed for the determination of $g(y, \lambda_1)$. We find

$$\begin{aligned}
(10.16)\quad \bar{g}(y, \lambda_1) &= .9221y - 3.4455y^3 + 5.7291y^5 - 5.9331y^7 + 4.46491y^9 \\
&\quad - 2.6252y^{11} + 1.2678y^{13} - .5186y^{15} + .1841y^{17} - .0577y^{19} \\
&\quad + .0162y^{21} - .0041y^{23} + .0010y^{25}. \\
\bar{L}\bar{g}(y, \lambda_1) &= 24.362y - 176.600y^3 + 478.351y^5 - 763.727y^7 \\
&\quad + 770.878y^9 - 602.030y^{11} + 372.596y^{13} - 190.113y^{15} \\
&\quad + 82.290y^{17} - 30.881y^{19} + 10.219y^{21} - 3.016y^{23} \\
&\quad + .798y^{25} - .186y^{27}.
\end{aligned}$$

Then in order to determine $g(y, \lambda_2)$ we assume $g^{(0)}(y, \lambda_2^{(0)}) = y - 2y^3 + y^5$ and write

$$g^{(1)}(y, \lambda_2^{(1)}) = \alpha_1^{(0)} g(y, \lambda_1) + g^{(0)}(y, \lambda_2^{(0)}).$$

We determine $\alpha_1^{(0)}$ from the relation

$$\int_0^1 g^{(0)}(y, \lambda_1)\overline{Lg}(y, \lambda_1)\, dy + \alpha_1 \int_0^1 g(y, \lambda_1)\overline{Lg}(y, \lambda_1)\, dy = 0$$

where $\int_0^1 g(y, \lambda_1)\overline{Lg}(y, \lambda_1)dy = .204$. From the differential equation we then obtain

$$g^{(2)}(y, \lambda_2^{(2)}) = \lambda_2^{(2)}\left[ay + by^3 + \int\int\int\int L\{g^{(1)}(y, \lambda_2^{(1)})\}\, dy\, dy\, dy\, dy\right]$$

where a and b are integration constants to be determined from the boundary conditions at $y = 1$. The approximation to λ_2 could be obtained from the condition $g_y(0, \lambda_2) = 1$. However, from the magnitude of the coefficients we note that the principal contribution to $g(y, \lambda_2)$ arises from terms other than the first and the aforementioned condition is not sensitive enough to fix the value of λ_2. Instead we use the following method. Let

$$g_n(y, \lambda_2^{(n)}) = a_1 y - a_3 y^3 + a_5 y^5 \cdots,$$

$$g_{n+1}(y, \lambda_2^{(n+1)}) = \lambda_2^{(n+1)}\{b_1 y - b_3 y^3 + b_5 y^5 \cdots\}.$$

If g_n were the solution we would have $g_{n+1} = g_n$ and hence $a_1 = \lambda_2^{(n+1)} b_1$; $a_3 = \lambda_2^{(n+1)} b_3$; $a_5 = \lambda_2^{(n+1)} b_5$; $\cdots$. It is therefore natural to take $\lambda_2^{(n+1)}$ as the arithmetical average of a_1/b_1, a_3/b_3, a_5/b_5, $\cdots$. Repeating this process until two successive iterates are the same we find

(10.17) $\lambda_2 = -86.2$

$$g(y, \lambda_2) = y - 43y^3 + 186y^5 - 398y^7 + 553y^9 - 546y^{11} + 407y^{13} - 236y^{15} + 109y^{17} - 41y^{19} + 12y^{21} - 3y^{23} + y^{25}.$$

$$\frac{d}{dy} g_\lambda(y, \lambda_2) \sim 200 \text{ for } y = \pm 1.$$

11. Evaluation of (9.9). We are now in position to evaluate the expansion (9.9). In order to compare our result with that obtained by Blasius' method, we choose $\varphi_1(x) = x$. Then (9.3) becomes

$$H(\lambda) = 2aU \int_0^\infty e^{-\lambda x} x\, dx = \frac{2aU}{\lambda^2} \tag{11.1}$$

so that the integral in (9.1) has a double pole at $\lambda = 0$. The residue due to this pole is

$$2aU \frac{d}{d\lambda}\left[\lambda^2 \frac{e^{\lambda x}}{\lambda^2} \frac{g(y, \lambda)}{\frac{d}{dy} g(y, \lambda) y = 1}\right]. \tag{11.2}$$

Differentiating we obtain

$$2aU\left[xe^{\lambda x}\frac{g(y,\lambda)}{g'(y,\lambda)}+\frac{e^{\lambda x}g_\lambda(y,\lambda)}{g'(y,\lambda)}-e^{\lambda x}g(y,\lambda)\frac{g'_\lambda(y,\lambda)}{[g'(y,\lambda)]^2}\right]_{y=1}. \tag{11.3}$$

For $\lambda = 0$, $(d/dy)g(y, \lambda) = 1$ and $g(y, \lambda) = \frac{1}{2}(y^3 - y)$. Also from (10.6) and (10.7) we find

$$g_\lambda(y,\lambda) = \frac{y}{84} - \frac{11}{120}y^3 + \frac{1}{60}y^5 - \frac{1}{420}y^7 .$$

Collecting our results we find that the expression for $\psi^1(x, y)$ becomes

$$\psi^1(x,y) = aU\left\{\frac{1}{42}y - \frac{11}{210}y^3 + \frac{1}{30}y^5 - \frac{1}{210}y^7\right\} + 2aU\{B_1(x)g(y,\lambda_1) + B_2(x)g(y,\lambda_2) + \cdots\}. \tag{11.5}$$

The explicit expression for (11.5) is unwieldy so we defer writing it out until a later stage. We have written only the terms arising from the poles λ_1 and λ_2. Actually, the contribution arising from the pole $\lambda_2 = -86.2$ is of the order 10^{-5}, or smaller, so that we neglect it this time. Examining the expression in (11.5) we note that the terms in the second brackets approach zero as x becomes large since $\lambda_1, \lambda_2, \cdots$ are all negative. Thus, if $x = \infty$ the resulting expression depends on y only. The condition $\psi^1(x, y) = 0$ for $x = 0$ is not satisfied by (11.5) because we have not accounted for all the poles $\lambda = \lambda_i$ of the integral in (9.9).

Since we have taken into consideration the contributions arising from the first two poles (11.5) will not be satisfactory for values of x near the mouth of the channel. However, the terms omitted contribute very little away from the entrance in comparison to that arising from $\lambda = \lambda_1$, and (11.5) can thus be used in that portion of the channel. When x is large the first expression in brackets of (11.5) will yield satisfactory results.

12. Solution of System III. We are now concerned with obtaining a solution of System III. In its present form the possibility of obtaining a solution would be quite difficult. However if we attempt a less ambitious program, and ask for a solution which shall be valid when x is large, we can make use of the foregoing. By analogy with (11.5) we ask for a solution which shall depend on y only when $x = \infty$. We therefore rewrite System III in the following form:

$$\begin{aligned} U\omega^2_{yy} - U(1-y^2)\omega^2_x - 2U\psi^2_x &= Ux\omega^1_{yy} + \omega^1_x\psi^1_y - \omega^1_y\psi^1_x \\ \omega^2 &= \psi^2_{yy} - 2ax\omega^1 - 4a^2Ux^2y \\ \psi^2 = 0, \qquad \psi^2_y &= 0 \text{ for } y = \pm 1. \end{aligned} \tag{12.1}$$

Since we are interested in a solution for which x is large, we do not try to satisfy the conditions for $x = 0$. Substituting for $\psi^1(x, y)$ its asymptotic expression, which is a function of y only, we obtain

$$\frac{\partial^4\psi^2}{\partial y^4} - (1-y^2)\frac{\partial^3\psi^2}{\partial x\partial y^2} - 2\frac{\partial\psi^2}{\partial x} = -2\frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial\psi^1}{\partial y}(x=\infty)\right\}. \tag{12.2}$$

Since $\psi^2(x, y)$ is to be a function of y only, the terms in the left member involving differentiation with respect to x, may be neglected and we get

$$\psi^2(y) = .01aU\{.19755378y - .56496943y^3 + .60317460y^5 - .30839002y^7 + .07936508y^9 - .00673401y^{11}\} \tag{12.3}$$

$$\frac{\partial^2 \psi^2}{\partial y^2} = .0069484\, aU \text{ for } y = +1.$$

Collecting our results for ψ^0, ψ^1 and ψ^2 we have as the solution for the stream function the following expression:

$$\begin{aligned}\psi(x, y) &= \psi^0(x, y) + \epsilon\psi^1(x, y) + \epsilon^2\psi^2(x, y) \\ &= aU\left(y - \tfrac{1}{3}y^3\right) + \epsilon aU\left\{\frac{y}{42} - \frac{11}{210}y^3 + \frac{1}{30}y^5 - \frac{1}{210}y^7\right\} \\ &\quad \frac{-2au\,\epsilon\, e^{-28.2210x}}{(28.2210)^2(.1059)}\{y - 3.01266y^3 + 3.78064y^5 \\ &\quad - 2.94517y^7 + 1.78945y^9 - .87886y^{11} + .36498y^{13} - .13081y^{15} \\ &\quad + .04134y^{17} - .01166y^{19} + .00297y^{21} - .00069y^{23} + .00014y^{25}\} \\ &\quad + \frac{aU\epsilon^2}{100}\{.19755378y - .56496943y^3 + .60317460y^5 \\ &\quad - .30839002y^7 + .07936508y^9 - .00673401y^{11}\}.\end{aligned} \tag{12.4}$$

13. Discussion of Backflow. Before proceeding with the discussion of backflow we shall compare our results with that obtained by the Blasius method. The first approximation $\psi^0(x, y)$ is the same in both instances. However, the expression for $\psi^1(x, y)$ are different. In the expression for $\psi^1(x, y)$ which we have found (see 11.5) there is an additional term $B_1(x)g(y, \lambda)$ where $g(y, \lambda)$ is given by (10.5). It is clear that as x increases the contribution of the term $B_1(x)g(y, \lambda_1)$ to $\psi^1(x, y)$ approaches zero. For values of x large enough $B_1(x)$ will be so small that the result obtained for $\psi^1(x, y)$ will reduce to that obtained by Blasius. Furthermore, the term $B_1(x)g(y, \lambda_1)$ contributes negatively to $\psi^1(x, y)$ so that as x becomes smaller the magnitude of $\psi^1(x, y)$ becomes smaller.

For backflow it is necessary that the stream function $\psi(x, y) = \psi^0 + \epsilon\psi^1 + \epsilon^2\psi^2$ shall have the same value along some stream line in the interior of the channel as it has on the boundary namely $\psi(x, y) = \psi(x, 1) = \frac{2}{3}aU$. Since $B_1(x)g(y, \lambda_1)$ increases (and consequently $\psi^1(x, y)$ decreases) as x decreases it is therefore necessary that ϵ shall assume increasingly larger values to give $\psi(x, y) = \frac{2}{3}aU$ (for a fixed y and decreasing value of x). Since the expression (11.5) represents an approximation to $\psi^1(x, y)$ which only accounted for the first two poles of the integral in (9.1) as said before, it will not satisfy the condition $\psi^1(0, y) = 0$ at $x = 0$. The value of $\psi^1(x, y)$ as given by (11.5) will be too large and the critical values of ϵ determined for points of detachment near the mouth of the channel as determined from (12.4) will be too small.

In view of the above reasoning it is likely that the point of detachment never reaches the mouth of the channel as ϵ increases but approaches the mouth as $\epsilon \to \infty$. This is indicated by the fact that all the exact functions $\psi^1, \psi^2, \cdots$ approach zero as $x \to 0$ so that, as the point of detachment moves toward the mouth, ϵ increases in order that we may obtain $\psi = \frac{2}{3}aU$. For $x = 0$ no finite value of ϵ gives backflow. This result cannot be deduced directly from the expression for $\psi(x, y)$ given in (12.4). However we shall later investigate the behavior of the flow in the vicinity of the mouth of the channel and show that backflow does not occur at the entrance.

In order to determine the condition for the backflow, we substitute the expressions for ψ^0_{yy}, ψ^1_{yy}, ψ^2_y into (6.1). Substituting we get

$$1 - \left\{\frac{8}{105} - .0524\, e^{-28.2210x}\right\}\epsilon - .003474\,\epsilon^2 = 0. \tag{13.1}$$

This relation enables us to fix both the position of the point of detachment and the slope of the boundary of the channel necessary for backflow. Although this relation is not valid for very small values of x it will be used to fix the point of detachment. The results obtained may still be useful for experimental purposes. The following table shows the relations mentioned.

POSITION OF BACKFLOW $x_B = \xi_B/aR$	$\epsilon x_B = \frac{m\xi_B}{a}$ (1)	$\epsilon = Rm/\varphi^1(0)$ (1)	$\epsilon = Rm/\varphi^1(0)$ (2)
0	0	15	42
.001	.014	14	21.9
.01	.13	13	17.4
.1	.94	9.4	13.7
∞	∞	9.24	13.1

(1) If $\psi = \psi^0 + \epsilon\psi^1 + \epsilon^2\psi^2$, (2) If $\psi = \psi^0 + \epsilon\psi^1$.

We have given the value of ϵ for $x_B = 0$ as determined from the expression (13.1). Actually, this is not the true value since (13.1) is not valid for $x = 0$ as explained above. Nevertheless, in the third column, we have given the critical value of ϵ as obtained by differentiating (12.4).

If we examine the results in the table we are able to obtain a comparison with the result obtained by Blasius. Since Blasius' result is independent of x it would appear that backflow takes place throughout the channel. We now see that in the Blasius case backflow takes place only at $x = \infty$. Actually the point of detachment may be situated at a finite distance from the mouth of the channel. The exact position of backflow depends on the slope of the channel and the Reynolds number. We have $\varphi = a(1 + x) = a + \epsilon R^{-1}\xi$ so that if we write $m = \epsilon R^{-1}$ we can conclude the dependence of the point of detachment $\xi = aRx_B$ on the Reynolds number. Since $\epsilon = mR$ is fixed, the larger the Reynolds number the smaller is the slope of the channel and the greater the distance of the point of detachment from the mouth of the channel.

It is interesting to note that the critical values of ϵ obtained from $\psi = \psi^0 + \epsilon\psi^1$ are not only larger but more sensitive to changes in x_B than the corresponding results obtained from $= \psi^0 + \epsilon\psi^1 + \epsilon^2\psi^2$. The improved values of ϵ change very little for large variations in x_B. The position of backflow moves away from the mouth of the channel quite rapidly with varying slope. This gives some indication of the difficulty encountered in obtaining experimental results. In the experiments conducted by Patterson [4] no attempt was made to locate the position of backflow. It is clear that it is no simple matter to find definite points of detachment. Actually, Patterson showed that there exists a range of values of ϵ for which backflow would take place thereby substantiating the Blasius theory.

14. Behavior near the Entrance to the Channel. Since our solution gives no insight into the behavior of the flow near the mouth of the channel, we shall develop two approximations to $\psi^1(x, y)$ valid for small values of x. Only the second of these will be valid when $y = 1$ and we shall try to determine the possibility of backflow near the mouth of the channel. The first approximation valid if $y \neq 1$ cannot be used for backflow. Our starting point is (8.2) which we write in the form

$$\psi^1(x, y) = f(x, y) + aUx(y - y^3) \tag{14.1}$$

where

$$f(x, y) = \frac{2aU}{2\pi i}\int_\gamma \frac{e^{\lambda x}}{\lambda^2} G(y\lambda)\, d\lambda \tag{14.2}$$

corresponding to the rectilinear boundary. We shall first show that if $y \neq 1$ and x is small, that the contribution from $f(x, y)$ may be neglected. Previously, we evaluated this integral on the assumption that x was large and the path of integration was moved to the left to $-\infty$. Since the eigenvalues were all situated on the negative real axis, we were able to use the theorem of residues in order to evaluate the integral approximately. However, if x is small, we must deform the path to $+\infty$. Any attempt to solve (9.4) as a power series in λ will fail since the solutions obtained are singular for $y = +1$. This suggests that we make the transformation

$$\zeta = -(6\lambda)^{1/3}(1 - y); \qquad h(\zeta) = (6\lambda)^{1/3}G(y) \tag{14.3}$$

with the provision that we shall let $\lambda \to \infty$ and take as our solution the one obtained from the limit problem. Thus (9.4), (9.5), (9.6), (9.7) become

$$\frac{d^4h}{d\zeta^4} = -\left\{\frac{1}{3}\zeta + \zeta(1296\lambda)^{-1/3}\right\}\frac{d^2h}{d\zeta^2} + 2(1296\lambda)^{-1/3}h \tag{14.4}$$

and

$$h(0) = 0; \qquad h'(0) = 1; \qquad h\{-(6\lambda)^{1/3}\} = 0, \qquad h''\{-(6\lambda)^{1/3}\} = 0. \tag{14.5}$$

We now let $\lambda \to \infty$ and obtain

$$d^3h/d\zeta^3 = A(\zeta); \qquad (d^3A/d\zeta^3) + \tfrac{1}{3}\zeta A = 0 \tag{14.6}$$

with the end conditions

$$h(0) = 0; \quad h'(0) = 1; \quad h(-\infty) = 0; \quad h''(-\infty) = 0. \tag{14.7}$$

One solution of the second order differential equation is given by Airy's integral

$$A(\zeta) = \frac{1}{2}\int_{-\infty}^{\infty} e^{it^3 - i\zeta t}\, dt \tag{14.8}$$

where the path of integration is taken along the real axis indented at the origin by a small circle in the lower half-plane. Since $A(-\infty) = 0$ one of the end conditions for $h(\zeta)$ is satisfied automatically by (14.8). The general solution of the problem is

$$h(\zeta) = C_1 + C_2\zeta + \tfrac{1}{2} C_3 \int_{-\infty}^{\infty} e^{it^3 - i\zeta t}\, dt/t^2$$

where C_1, C_2 are real, C_3 is complex and are to be chosen so that the end conditions are satisfied. Since $h(0) = 0$; $h'(0) = 1$ we must have

$$C_1 = -\tfrac{1}{2} C_3 \int_{-\infty}^{\infty} e^{it^3}\, dt/t^2; \qquad C_2 - \tfrac{1}{2} i\, C_3 \int_{-\infty}^{\infty} e^{it^3}\, dt/t = 1.$$

If we make the substitution $t^3 = s$ in the second integral, we obtain

$$-\frac{1}{2} i \int_{-\infty}^{\infty} e^{it^3}\, dt/t = \frac{1}{3}\int_0^{\infty} \sin s\, ds/s = \pi/6$$

so that $C_2 + \pi/6 C_3 = 1$. Now, consider the integral

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{it^3}\, dt/t^2$$

whose integrand has no singularities in the lower half-plane. The original path may be deformed into the one shown in fig. 1. Along I, the integral is $\frac{1}{2}\int_{\tau}^{-\tau - i\sigma} e^{it^3}\, dt/t^2$ where $t = -\tau + is$ and τ is fixed. Since $|e^{it^3}/t^2| \leq 1/(\tau^2 + s^2)$ the integral is smaller than $1/2\tau$. Thus as $\tau \to \infty$ the contribution from I vanishes. The contribution from II and III can be shown to vanish in a similar manner. Therefore $\int_{-\infty}^{\infty} e^{it^3}\, dt/t^2 = 0$ and hence $C_1 = 0$. According to (14.7) we should have $h(-\infty) = 0$ for $\zeta = -\infty$. Thus

$$C_2\zeta + \tfrac{1}{2} C_3 \int_{-\infty}^{\infty} e^{it^3 - i\zeta t}\, dt/t^2 = 0 \quad \text{for } \zeta = -\infty.$$

Consequently we must determine the asymptotic behavior of the integral

$$I(\zeta) = \frac{1}{2}\int_{-\infty}^{\infty} e^{it^3 - i\zeta t}\, dt/t^2 \quad (\zeta < 0)$$

as $\zeta \to -\infty$. We put $-\zeta = 3\sigma^{2/3}$, $t = \sigma^{1/3} z$ and obtain

$$I(\zeta) = \tfrac{1}{2}\sigma^{-1/3}\int_{-\infty}^{\infty} e^{\sigma w(z)}\, dz/z^2$$

where $w(z) = i(z^3 + 3z)$. To evaluate the integral we shall use the method of steepest descents. For the purposes of our development, it will be sufficient to determine the first term of the asymptotic expansion.

Since $w'(z) = 3(1 + z^2)$ vanishes at the points $z = \pm i$ these points are the cols of $w(z)$. This path is to be deformed in such a manner that $I(w) =$ constant and $R(w) < 0$ on the new path. By Cauchy's theorem we know that the value of the integral will remain unchanged if no singularities are included during the process of deformation. However since the integrand has a pole at $z = 0$ and the new path will pass through $z = i$, we must take account of the pole in changing the path of integration. Consequently consider

$$w(z) = u + iv = (y^3 - 3yx^2 - 3y) + i(x^3 - 3xy^2 + 3x).$$

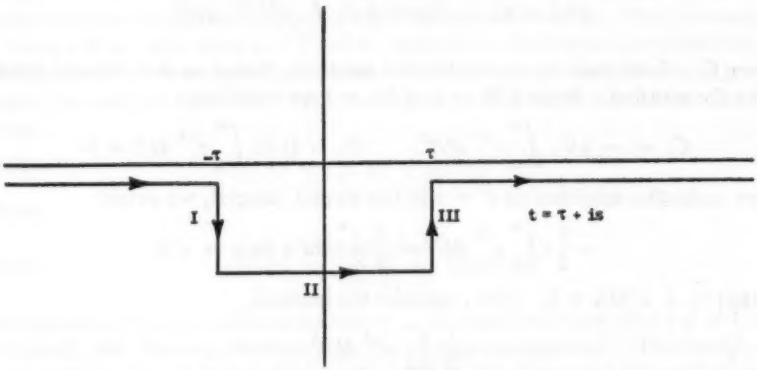

FIG. 1. PATH OF INTEGRAL IN t-PLANE

The path desired is given by $v(x, y) = v(0, \pm 1) = 0$. $v(x, y) = 0$ yields the line $x = 0$ and the hyperbola $x^2 - 3y^2 + 3 = 0$ which intersect in the cols. The real part is $u(x, y) = y^3 - 3yx^2 - 3y$ and $u(x, y) = 0$ on the line $y = 0$ and the hyperbola $y^3 - 3x^2 - 3 = 0$. The hyperbola $x^2 - 3y^2 + 4 = 0, y > 0$ is the path of steepest descent. By Cauchy's theorem we deform the original path into this hyperbola. In so doing we must account for the residue due to the pole at $z = 0$. Thus our integral becomes

$$I(\sigma) = \tfrac{1}{2}\sigma^{-1/3}\left\{\int_{(-\sqrt{3}+i)\infty}^{i} + \int_{i}^{(\sqrt{3}+i)\infty}\right\} + \pi i \operatorname{Res}_{z=0}$$

The contribution from the residue is

$$\frac{\pi i}{2\sigma^{1/3}}\frac{d}{dz}\{e^{\sigma w(z)}\} = \frac{\pi}{2}\zeta$$

In order to evaluate the integrals in the brackets we note that the values of $u = y^3 - 3yx^2 - 3y$ along the hyperbola $x^2 - 3y^2 + 3 = 0$ have a maximum at the col, $z = i$. Since $x^2 = 3y^2 - 3$ we have $u = -8y^3 + 6y$. When $y = 1, u = -2$ so that if $y = 1 + \alpha, \alpha > 0$ (i.e. another point on $x^2 = 3y^2 - 3$) we have $u =$

$-2 - 18\alpha - 24\alpha^2 - 8\alpha^3$ which is smaller than $u = -2$. Thus the maximum contribution to the integral occurs in the neighborhood of the col. In this neighborhood we may replace the hyperbolic path by a segment from $-\delta$ to $+\delta$. This segment is parallel to the real axis and therefore has the same direction as the hyperbola at $z = +i$. We then have the following approximation as the principal part of the asymptotic expansion [5].

$$\int e^{\sigma w(z)}\,\phi(z)\,dz \sim e^{\sigma w(z_0)}\,\phi(z_0)\left\{\frac{-2\pi}{\sigma w''(z_0)}\right\}^{\frac{1}{2}}$$

where z_0 is the col of $w(z)$. Employing this relation we ultimately obtain

$$I(\zeta) \sim \frac{\pi}{2}\zeta - \tfrac{1}{2}\sqrt{\frac{\pi}{3}}\left(-\frac{\zeta}{3}\right)^{-5/4} exp\left[-2\,(-\zeta/3)^{3/2}\right].$$

The condition $h(-\infty) = 0$ takes the form $C_2\zeta + C_3 I(\zeta) \to 0$ as $\zeta \to -\infty$ or $\left(C_2 + \frac{\pi}{2}C_3\right)\zeta \to 0$ as $\zeta \to -\infty$ whence $C_2 + \frac{\pi}{2}C_3 = 0$. This together with $C_2 + \frac{\pi}{6}C_3 = 1$ yields $C_2 = 3/2$ and $C_3 = {}^{-3}/_{\pi}$. We have thus found that

$$h(\zeta) = \frac{3}{2}\zeta - \frac{3}{\pi}I(\zeta)$$

and asymptotically

$$h(\zeta) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\left(\frac{\zeta}{3}\right)^{-5/4} e^{-2(-\zeta/3)^{3/2}}.$$

The value of $G(y, \lambda)$ for a given value of $y \neq 1$ and for sufficiently large λ dependent on y is given by

$$G(y,\lambda) = \frac{\beta_2}{\lambda^{3/4}}\,e^{-\beta_1\lambda^{1/2}} \qquad \begin{cases} \beta_1 = 2\sqrt{6}\left(\dfrac{1-y}{3}\right)^{3/2} \\ \beta_2 = \sqrt{\dfrac{3}{\pi}}\,\dfrac{9U}{6^{3/4}\left(\dfrac{1-y}{3}\right)^{5/4}} \end{cases}$$

so that, if λ is large, we have for $y \neq 1$

$$f(x,y) = \frac{\beta_2}{2\pi i}\int_{\gamma} e^{\lambda x - \beta_1\lambda^{1/2}}\,\frac{d\lambda}{\lambda^{11/4}} \tag{14.9}$$

the path of integration being a line parallel to the imaginary axis in the right half-plane. To evaluate this integral we make use of the fact that x is small. We first make the substitution $\sqrt{\lambda x} = \sqrt{\sigma}t$ where $\sigma x = \beta_1^2$. Thus (14.9) becomes

$$f(x,y) = \frac{2\beta_2}{2\pi i}\left(\frac{x}{\beta_1}\right)^{7/2}\int \exp\,\{\sigma(t^2 - t)\}\,dt/t^{9/2} \tag{14.10}$$

where σ is large when x is small. We set $w(t) = t^2 - t$ then $w'(\frac{1}{2}) = 0$. We wish to deform the path in the t plane in such a manner that $I(w) =$ constant, $R(w) < 0$. Thus setting $t = s + i\tau$, $w = u + iv$ then $u = s^2 - s - \tau^2$; $v = 2s\tau - \tau$.

Since $u < 0$ between the two branches of the hyperbola $s^2 - s - \tau^2 = u$ the line $s = \frac{1}{2}$ in this region, the desired path is this line. By Cauchy's theorem we may deform the hyperbolic path into the rectilinear path $s = \frac{1}{2}$. Thus, if we put $t = \frac{1}{2} + i\tau$ in (14.10) we obtain

$$f(x, y) = \frac{\beta_2}{\pi}\left(\frac{x}{\beta_1}\right)^{7/2} \frac{e^{-\sigma/4}}{2^{9/2}} \int_{-\infty}^{\infty} e^{-\sigma\tau^2} (1 + 2it)^{-9/2}\, dt. \tag{14.11}$$

If σ is large enough the bulk of the integral is contributed when $|2\tau| \leq 1$. Thus we may expand the factor $(1 + 2i\tau)^{-9/2}$ and integrate termwise to obtain an asymptotic expansion in τ. Keeping the first two terms of this expansion we obtain

$$f(x, y) = \frac{243\sqrt{3}}{4096\pi}\, aU\, \frac{x^4}{(1-y)^8}\left\{1 - \frac{891x}{32(1-y)^3} + \cdots\right\} \cdot exp\left[\frac{-2(1-y)^3}{9x}\right]. \tag{14.12}$$

We thus have from (14.1) and (14.12) as our approximation to $\psi^1(x, y)$ when x is small and $y \not\equiv 1$ the expression

$$\psi'(x, y) = aUx(y - y^3) + \frac{243\sqrt{3}}{4096\pi}\, \frac{aUx^4}{(1-y)^8}\, exp\left[\frac{-2(1-y)^3}{9x}\right]. \tag{14.13}$$

It is clear that if $y \not\equiv 1$ is given the second term is small as compared with the first. In the region near $y = 1$ this term has no meaning so that we may take as our approximation the expression

$$\psi^1(x, y) = aUx(y - y^3) \tag{14.14}$$

with the proviso that it shall not be used in the neighborhood of $y = 1$. The relation (14.14) may be used in conjunction with ψ^0 to account for the region near the mouth of the channel but not near $y = 1$. We have stated previously that (12.5) is not valid when x is small. By joining the two solutions in a continuous fashion from the tabular values obtained by computation we can draw the stream lines $\psi(x, y)$ which describe the flow.

15. Behavior near the Boundary at the Entrance. Since the expression in (14.14) is not to be used when $y = 1$ it cannot be used to find a point of detachment of the flow. Actually no backflow condition (i.e. a value of $(\epsilon > 0)$ can be found. It is thus natural to ask for another expression which shall approximate $\psi^1(x, y)$ in the neighborhood of $y = 1$. This is equivalent to finding a solution of (14.6) and (14.7) when ζ is small. The general solution of $A'' + \frac{1}{3}\zeta A = 0$ is

$$A(\zeta) = A_1\left\{1 - \frac{\zeta^3}{2\cdot 3\cdot 3} + \cdots\right\} + A_2\zeta\left\{1 - \frac{\zeta^3}{3\cdot 4\cdot 3} + \cdots\right\} \tag{15.1}$$

Thus when ζ is small we have

$$h(\zeta) = a + b\zeta + 0(\zeta^3) \tag{15.2}$$

and since $h(0) = 0$, $h'(0) = 1$ we find $a = 0$, $b = 1$. Neglecting powers of ζ higher than the first, our approximation is

(15.3) $$h(\zeta) = \zeta$$

From (14.3) we then find

(15.4) $$G(y, \lambda) = -(1 - y)$$

Therefore (14.2) becomes

(15.5) $$f(x, y) = \frac{-2aU(1 - y)}{2\pi i} \int_{\Gamma} \frac{e^{\lambda x}}{\lambda^2} d\lambda = -2aUx(1 - y)$$

Thus when x is small and y is near $y = 1$ we have as the approximation to $\psi^1(x, y)$.

(15.6) $$\psi^1(x, y) = Uax\{(y - y^3) - 2(1 - y)\} = -Uax(1 - y)^2(y + 2)$$

This solution and (14.14) describe the flow in the entrant portion of the channel. However when (15.6) is used to determine backflow no such criterion can be fixed. Yet we are able to draw the following important conclusion. We have

$$\psi = \psi^0 + \epsilon\psi^1$$

or

(15.7) $$\psi = Ua(y - \tfrac{1}{3}y^3) - \epsilon aUx(1 - y)^2(y + 2)$$

Since the Poiseille distribution describes the flow at the entrance, we have $u = \partial\psi/\partial y = 0$ for $x = 0$ when $y = 1$. Furthermore from (15.7) we see that $u = \partial\psi/\partial y = 0$ for $y = 1$ when x is small. This result has been found to determine the behavior at the boundary near the mouth of the channel. From our earlier results no conclusion could be drawn about the flow in this portion of the channel. Where x was not too small, a value of ϵ was found for which a point of detachment was obtained. However, once ϵ exceeded a certain critical value, the point of detachment x_s was so near the entrance to the channel that the expression (12.5) for $\psi(x, y)$ was no longer valid. However, by essentially qualitative statements it was possible to conclude that backflow could not occur at the mouth. The expression given in (15.7) tends to verify that result. The second part of (15.7) is essentially positive since $x > 0$ and $0 < y < 1$. Since it contributed negatively to $\psi^0 = aU(y - \frac{1}{3}y^3)$ it is impossible to obtain a streamline $\psi = \frac{2}{3}aU$ in this portion of the channel and consequently backflow cannot occur at the entrance.

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The Computation Laboratory
National Bureau of Standards

(Received July 6, 1948)

SYNTHESIS OF RC-NETWORKS[1]

By E. A. Guillemin

Introduction. For various practical reasons one is sometimes confronted with the problem of meeting arbitrarily specified transfer characteristics in a network which contains no inductive elements. It has for some time been known that a solution to this problem is possible through the use of vacuum tubes in conjunction with the passive RC-elements in a circuit employing the principles of feedback. While the absence of inductive elements may in this way be completely compensated for, one achieves this result at the expense of having a network whose frequency characteristic is dependent upon the constancy and life of the vacuum tubes and their power supplies. Where ruggedness and permanence are desirable, one prefers a network whose frequency response characteristic is dependent upon passive elements only. The question thus arises as to what, if any, are the limitations of passive networks containing only R's and C's.

The following discussion shows that, except for a constant loss, any transfer characteristic may be approximated arbitrarily closely by means of a linear passive network containing only R's and C's, and its physical realization is always possible in an unbalanced form suitable for operation between conventional vacuum tubes.[2] Since resistances may be regarded as coils whose "Q" 's are zero, one is able to infer that any desired response characteristics may be obtained from a network with arbitrarily poor quality coils, or that a high quality of performance on the part of a network does not necessarily require high quality coils, provided only that one can compensate for constant loss.

The discussion logically divides into several parts: First, a consideration of the form of the transfer function, pointing out its limitations as contrasted to one representing a passive RLC-network; second, a discussion showing how a function having such limitations may be found to approximate arbitrarily closely to a specified characteristic; third, a discussion of how a transfer function thus found may be realized by a network (that is, the synthesis procedure); and lastly a presentation of some illustrative examples.

[1] This study was begun in the Spring of 1944, being suggested by certain phases of the M. I. T. Radiation Laboratory program. Although the theoretical synthesis procedure was completed before the close of the war, numerous detailed questions essential to the determination of practical networks were left unanswered. These were later supplied through the ingenuity of Commanders K. W. Patrick and N. E. Thomas who carried the work to a successful and satisfying conclusion, presenting these contributions in their joint Master's Thesis entitled: "A Design Procedure for Linear Passive RC-Filter Networks," May 1946.

[2] It should be pointed out here that there are some limitations on the generality of phase characteristics obtainable with RC-networks. The minimum phase associated with any amplitude characteristic is, of course, always obtainable and some additional phase shift may also be had, but just what the limitations on the phase characteristic are can at present not be expressed in simple terms.

The transfer function of an RC-network. Consider first a two terminal-pair network (Fig. 1) by itself; that is, apart from any terminal impedances. The terminal voltages and currents may be related through the equations

$$I_1 = y_{11}E_1 + y_{12}E_2\,, \qquad I_2 = y_{21}E_1 + y_{22}E_2\,, \tag{1}$$

in which y_{11}, y_{22}, and $y_{12} = y_{21}$ are the familiar short-circuit driving-point and transfer admittances. If the network N contains only R's and C's then y_{11} and y_{22} are rational functions of the complex frequency variable $\lambda = \sigma + j\omega$ having only simple zeros and poles alternating on the negative real axis of the λ-plane. Moreover, the lowest (or smallest) critical frequency of y_{11} or of y_{22} must be a zero. These conditions completely fix the character of either y_{11} or y_{22}.

In order to investigate the character of the short-circuit *transfer* function y_{12}

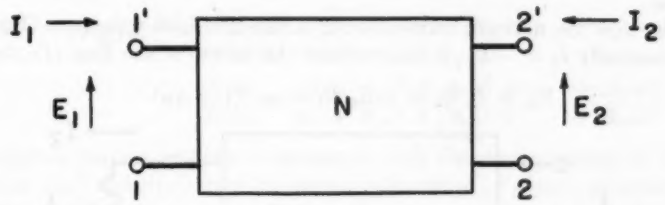


FIG. 1

one observes (according to the work of Brune and Gewertz) that the function (quadratic form)

$$y = y_{11}x_1^2 + 2y_{12}x_1x_2 + y_{22}x_2^2 \tag{2}$$

must have the same properties as either y_{11} or y_{22} for all real values of x_1 and x_2. It follows, therefore, that y_{12} can have only simple poles restricted to the negative real axis (as have the functions y_{11} and y_{22}), but the zeros of y_{12} are left arbitrary.

From the stated properties of y_{11} or y_{22} one may observe that these functions are quotients of polynomials in which the degree of the numerator is either equal to or greater by one than the degree of the denominator. In the function y_{12}, on the other hand, the degree of the numerator has the same upper limit but is otherwise arbitrary.

If, in any of the y's, the degree of the numerator exceeds that of the denominator, then that function has a pole at $\lambda = \infty$. In the functions y_{11}, y_{22}, and y the residue in this pole must be real and positive; in y_{12} this residue must be real but need not be positive. At all other poles the residues of y_{11}, y_{22}, and y must be real and negative while again that of y_{12} may have either sign. Denoting such residues by the letter k with subscripts identical to those on the corresponding y's, Eq. 2 yields the relation

$$k = k_{11}x_1^2 + 2k_{12}x_1x_2 + k_{22}x_2^2\,. \tag{3}$$

If k as well as k_{11} and k_{22} is to remain positive (resp. negative) for all possible real values of x_1 and x_2, it is necessary and sufficient that

$$k_{11}k_{22} - k_{12}^2 \geqq 0, \tag{4}$$

whence it is clear that if k_{12} is nonzero then *both* k_{11} and k_{22} must likewise be nonzero. Therefore, if at some value of λ either y_{11} or y_{22} or both have a pole, then the function y_{12} may or may not also contain this pole, but if y_{12} has a pole at some point then both y_{11} and y_{22} must have poles there.

The relation 4 is referred to as the *residue condition*. It applies also to the residues of impedances, and holds for any network involving two kinds of elements (*RL*, *RC*, and *LC* cases). It has many uses. At the moment it yields the conclusion that if the degree of the numerator in y_{12} exceeds that of the denominator (by not more than one, of course) then the same must be true of the functions y_{11} and y_{22}.

Consider now the network terminated in a one-ohm load resistance (Fig. 2). Since numerically $I_2 = -E_2$, it follows from the second of the Eqs. (1), that

$$Y_{12} = I_2/E_1 = -E_2/E_1 = y_{12}/(1 + y_{22}). \tag{5}$$

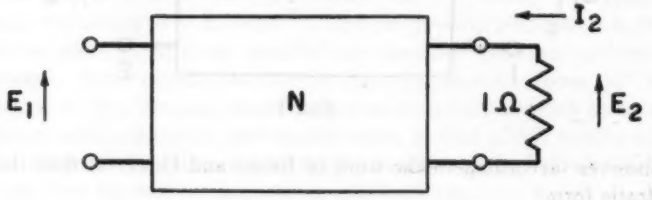


FIG. 2

The function Y_{12} may be regarded as the desired transfer function under load. The quantity $1 + y_{22}$ in the denominator of Y_{12} is the driving-point admittance of an *RC*-network (namely, that for the admittance y_{22} shunted by a conductance of one mho.). Hence the zeros of $1 + y_{22}$ (and, therefore, the poles of Y_{12}) must be simple and are restricted to the negative real axis of the λ-plane. The zeros of Y_{12} are either zeros of y_{12} or poles of y_{22} not also contained in y_{12}. As is shown later on, the synthesis procedure is so carried out that y_{12} and y_{22} have identical simple poles. Thus Y_{12} has zeros only where y_{12} has them, and these may have any multiplicity and may lie anywhere in the complex λ-plane.

Thus the transfer function Y_{12} of an *RC*-network terminated in a resistive load is discovered to be a quotient of polynomials with real coefficients such that the degree of the numerator does not exceed that of the denominator, and the denominator has simple zeros restricted to the negative real axis while the numerator may have zeros anywhere.

If the network N is permitted to contain inductive elements as well as resistive and capacitive ones, it is found that the transfer function Y_{12} is more general only in that the zeros of the denominator are restricted to the left half of the λ-plane (instead of to the negative real axis) and they need not be simple. Nevertheless,

this added generality gives the *RLC*-network considerable practical advantages, although as the following discussion shows, one may approximate equally well to any desired transfer characteristic with an *RC*-network, except for an additional constant loss.

The approximation problem and its solution. According to the preceding discussion the transfer function Y_{12} may be written in the form

$$Y_{12}(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n}{b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_n\lambda^n}. \tag{6}$$

The desired transfer characteristic is usually specified by prescribing graphically the absolute value of Y_{12} as a function of real frequencies ($\lambda = j\omega$). Since for $\lambda = j\omega$, $p(-\lambda)$ is the conjugate of $p(\lambda)$, it is clear that

$$\begin{aligned} |Y_{12}(j\omega)|^2 &= \left\{\frac{p(\lambda)\cdot p(-\lambda)}{q(\lambda)\cdot q(-\lambda)}\right\}_{\lambda=j\omega} \\ &= \frac{A(\omega^2)}{B(\omega^2)} = \frac{A_0 + A_1\omega^2 + A_2\omega^4 + \cdots + A_n\omega^{2n}}{B_0 + B_1\omega^2 + B_2\omega^4 + \cdots + B_n\omega^{2n}}. \end{aligned} \tag{7}$$

The approximation problem is concerned with the determination of the polynomials $A(\omega^2)$ and $B(\omega^2)$ from the desired behavior of $|Y_{12}(j\omega)|$, as is discussed in detail presently. Once the coefficients $A_0 \cdots A_n$ and $B_0 \cdots B_n$ are known, one may obtain the entire transfer admittance, as given in Eq. (6), by the method of Gewertz. Beginning with the denominator in Eq. (7) it is observed that $q(\lambda)\cdot q(-\lambda) = B(-\lambda^2)$ since $\lambda = j\omega$ implies $\omega^2 = -\lambda^2$. Now the zeros of $q(-\lambda)$ are those of $q(\lambda)$ with reversed sign. The zeros of $q(\lambda)$ being all negative real, it is clear that those of $q(-\lambda)$ are all positive real, and that the polynomial $B(-\lambda^2)$ has both sets of zeros.

The procedure for finding $q(\lambda)$ from $B(\omega^2)$ is perfectly clear. Through setting $\omega^2 = -\lambda^2$ one obtains $B(-\lambda^2)$, and having this polynomial one may find all of its zeros which must, of course, occur in pairs of real values with opposite signs. The polynomial $q(\lambda)$ is then formed from the negative set of zeros, being uniquely determined by these except for a multiplying constant which is of no importance.

It should be observed at this point that the ω^2-roots of the equation $B(\omega^2) = 0$ should be negative real and simple since the λ^2-roots must be positive real and simple in order that the λ-roots may be real and distinct and occur in pairs with opposite signs. This restriction on the denominator polynomial in Eq. (7) is important, for it is the consequence of specifying that the resultant network shall involve only R's and C's.

The determination of $p(\lambda)$ from $A(\omega^2)$ follows a similar pattern, differing only in that the zeros of $p(\lambda)$ are completely unrestricted so that, from the zeros of $A(-\lambda^2)$ which collectively form a pattern having complete symmetry about the real and imaginary axes of the λ-plane, it is not possible to assign a specific group as uniquely characterizing $p(\lambda)$ unless one invokes the stipulation that the network be of the minimum phase-shift variety or makes some other specification

fixing the allocation of zeros between the polynomials $p(\lambda)$ and $p(-\lambda)$. The minimum phase-shift requirement is here accepted as the determining factor since $p(\lambda)$ then becomes a Hurwitz polynomial with coefficients $a_0 \cdots a_n$ which are surely positive, a circumstance that is essential to the successful accomplishment of the synthesis procedure to be described later on. Thus the polynomial $p(\lambda)$ is determined except for a constant multiplier by assigning to it those zeros of $A(-\lambda^2)$ lying in the left half of the λ-plane.

Attention is now turned to the problem of determining the coefficients in $A(\omega^2)$ and $B(\omega^2)$ so that the expression given by Eq. 7 approximates a specified characteristic with stipulated tolerances. The solution to this problem may be carried out in a variety of ways.

Consider the function

$$V_n(x) = \cos(n \cos^{-1} x), \tag{8}$$

which, except for the factor $1/2^{n-1}$ is identical with the Tschebyscheff polynomial of the first kind $T_n(x)$. Its detailed properties need, therefore, not be discussed here, except to recall that it is a polynomial in x of the degree n. If one lets

$$x = \cos \phi, \tag{9}$$

then

$$V_n(x) = \cos n\phi, \tag{10}$$

and it becomes clear that the trigonometric polynomial in ϕ

$$f(\phi) = \alpha_0 + \alpha_1 \cos \phi + \alpha_2 \cos 2\phi + \cdots + \alpha_n \cos n\phi, \tag{11}$$

is equivalent to the ordinary polynomial in x

$$g(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_n x^n. \tag{12}$$

It remains to relate x to ω in such a way that the interval $-1 < x < 1$ corresponds to the interval $-\infty < \omega < \infty$. This step may be accomplished through several functions, a particularly effective one being

$$x = (1 - \omega^2)/(1 + \omega^2), \tag{13}$$

since with Eq. 9 it yields

$$\omega = \tan \phi/2 \tag{14}$$

as the relation between ϕ and ω. Thus the range $-\pi < \phi < \pi$ corresponds to $-\infty < \omega < \infty$ with $\phi = 0$ equivalent to $\omega = 0$ and $\phi = \pm\pi/2$ to $\omega = \pm 1$.

A graphically or analytically specified function of ω may thus readily be transferred to the ϕ-domain where, through established methods, the coefficients $\alpha_0 \cdots \alpha_n$ of a trigonometric polynomial (Eq. 11) affording the desired approximation may be found. Using Eq. 10, this result is converted into an ordinary polynomial in x (Eq. 12), and finally through the transformation 13 one obtains a function of the form

$$F(\omega^2) = \frac{A_0 + A_1\omega^2 + A_2\omega^4 + \cdots + A_n\omega^{2n}}{(1 + \omega^2)^n} \tag{15}$$

which approximates the originally specified function of ω with the same tolerances as does the trigonometric polynomial 11 in the ϕ-domain.

Reference to Eq. (7) and the preceding discussion shows that this result is not acceptable, for the denominator has an nth order, zero at $\omega^2 = -1$. A cure for this difficulty is easily found. One begins by choosing the polynomial $B(\omega^2)$ in the enominator of Eq. (7) according to what appears best to suit the subsequent synthesis procedure (as will be mentioned again later on). Thus $B(\omega^2)$ may be formed through choosing a set of negative real ω^2-roots of the equation $B(\omega^2) = 0$. If these are chosen in pairs having the geometric mean value -1, then the corresponding trigonometric polynomial is symmetrical about $\phi = \pi/2$ a useful condition in the design of low- or high-pass filters with cutoff at the nominal value $\omega = 1$. According to the preceding discussion, one may then associate a trigonometric polynomial $f_2(\phi)$ with the function

$$F_2(\omega^2) = \frac{B(\omega^2)}{(1+\omega^2)^n} = \frac{B_0 + B_1\omega^2 + B_2\omega^4 + \cdots + B_n\omega^{2n}}{(1+\omega^2)^n}. \tag{16}$$

The specified function of ω, transferred to the ϕ-domain, is $f(\phi)$ and may be represented as

$$f(\phi) = f_1(\phi)/f_2(\phi), \tag{17}$$

so that

$$f_1(\phi) = f(\phi)\cdot f_2(\phi). \tag{18}$$

Since $f(\phi)$ and $f_2(\phi)$ are known, a graph of their product is readily obtained in the ϕ-domain and approximated by a trigonometric polynomial $f_1(\phi)$ so as to control the tolerance with which the quotient f_1/f_2 approximates $f(\phi)$. From $f_1(\phi)$ there is found a function $F_1(\omega^2)$ having the form given in Eq. 15, whence through division by $F_2(\omega^2)$ one obtains the function

$$F(\omega^2) = \frac{F_1(\omega^2)}{F_2(\omega^2)} = \frac{A_0 + A_1\omega^2 + \cdots + A_n\omega^{2n}}{B_0 + B_1\omega^2 + \cdots + B_n\omega^{2n}} \tag{19}$$

approximating the desired characteristic in the ω-domain with the same tolerances that f_1/f_2 approximates $f(\phi)$ in the ϕ-domain.

Since the functions F_1 and F_2 have the same degree, $f_1(\phi)$ and $f_2(\phi)$ are trigonometric polynomials having the same number of terms. Therefore, a choice for the number of terms in $f_2(\phi)$ having been decided upon at the outset, one is no longer free to choose the number of terms in $f_1(\phi)$ needed to satisfactorily approximate $f(\phi)\cdot f_2(\phi)$ in Eq. 18. This circumstance is a disadvantage in dealing with the approximation problem since the number of terms needed for a desired quality of approximation does not become evident until after this choice has been decided upon. Some experience with the method is, therefore, essential to its successful use.

The approximation problem is thus solved formally, and the method of its solution shows that any characteristic that may arise in connection with practical problems can be approximated with any needed tolerances.

Network realization of the transfer function. In the expression for $Y_{12}(\lambda)$ as given by Eq. (6), let the denominator polynomial $q(\lambda)$ be separated into two additive parts

$$q(\lambda) = q_1(\lambda) + q_2(\lambda), \tag{20}$$

whereupon one may manipulate this expression as follows

$$Y_{12}(\lambda) = \frac{p}{q_1 + q_2} = \frac{p/q_1}{1 + q_2/q_1}. \tag{21}$$

Comparison with Eq. 5 then suggests the identifications

$$y_{12} = p(\lambda)/q_1(\lambda); \qquad y_{22} = q_2(\lambda)/q_1(\lambda). \tag{22}$$

The separation of $q(\lambda)$ into parts must, of course, be done in such a way that the ratio q_2/q_1 is the physically realizable driving-point admittance of an *RC*-network. This point is further elaborated upon below.

If the polynomial $p(\lambda)$ in y_{12} is now thought of as completely written out as in Eq. 6, it is seen that y_{12} may be expressed in the form

$$y_{12} = \frac{a_0}{q_1} + \frac{a_1\lambda}{q_1} + \frac{a_2\lambda^2}{q_1} + \cdots + \frac{a_n\lambda^n}{q_1}. \tag{23}$$

It is well known that y_{22} may be realized in the form of a large variety of ladder networks with series and shunt branches that are either resistance or capacitance elements, and a little thought reveals that such ladder networks can be made to have short-circuit transfer admittances y_{12} given by any one of the separate terms in the expression (23). This fact suggests that the desired synthesis may be achieved through a parallel connection of a set of such ladder networks, individually multiplied by appropriate scale factors so as to obtain a resultant structure still having the admittance y_{22}.

The obvious requirement that for any component network the functions y_{22} and y_{12} should have identical poles may be seen to be essentially fulfilled in all cases. Visualize a network with two terminal-pairs and give to the poles of y_{22} and y_{12} the following physical interpretation: Suppose that an energy source of some sort is momentarily inserted at the terminal-pair 2-2′ (Fig. 1), and that upon its removal this terminal-pair is left short-circuited (the terminal-pair 1-1 is shorted also). Suppose first that the transient current at the shorted end 2-2′ is of interest. This transient current is given by a sum of exponential terms (in general occurring in pairs of conjugate complex values) in which each pair represents a complex natural frequency (or frequency and damping factor) of the network with its applied short-circuit constraints. These frequencies are the poles of y_{22}, and the so-called characteristic equation yielding them is $q_1(\lambda) = 0$. Now if, under the same identical conditions, one asks for the transient current at the short-circuited terminal-pair 1-1′ it is clear that the same natural frequencies are involved and are given by the same characteristic equation because the physical system is the same. Therefore the poles of y_{12} are the same as those of y_{22} (and of y_{11} too for that matter), except for the following consideration. If the ter-

minal-pair 2-2′ is bridged (that is, shunted) by any two-terminal network, then, since a short-circuit constraint is applied at that terminal-pair, the poles of the admittance of this two-terminal network, while contained in y_{22}, clearly cannot be poles of y_{12} also because the transient current at the shorted end 1-1′ is unaffected by the presence or absence of the two-terminal network.

In order to secure identical poles for y_{22} and y_{12}, therefore, it is sufficient to require that the network begin with a series branch at the end 2-2′ unless that branch is a pure resistance. This condition is observed in the detailed formulation of the synthesis procedure to be described presently.

A few remarks are in order regarding the separation of the polynomial $q(\lambda)$, as indicated in Eq. 20, so that y_{22} in Eq. (22) becomes the realizable admittance of an *RC*-network. Since the immediately preceding discussion shows that the net-

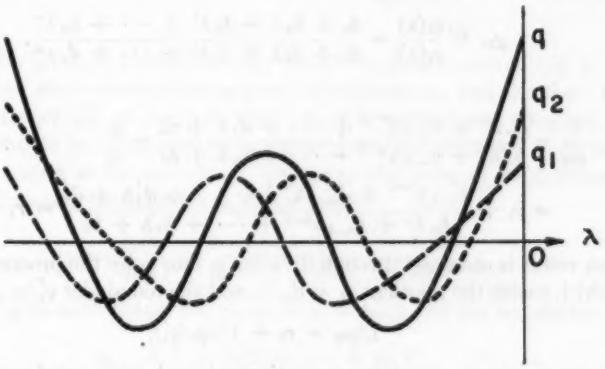


FIG. 3

work realizing y_{22} should not begin with a shunt capacitance branch, it is, moreover, necessary that y_{22} should have no pole at $\lambda = \infty$. The latter condition is met by separating $q(\lambda)$ in such a way that $q_1(\lambda)$ and $q_2(\lambda)$ have the same degree.

The proper method of separation is indicated in Fig. 3 where $q(\lambda)$, $q_1(\lambda)$, and $q_2(\lambda)$ are shown plotted versus real values of λ. It is clear that if $q_1(\lambda)$ and $q_2(\lambda)$ are any two polynomials whose simple zeros alternate, then their sum $q(\lambda)$ has zeros that lie between pairs of zeros belonging respectively to $q_1(\lambda)$ and $q_2(\lambda)$. The method of separating $q(\lambda)$ so as to achieve the desired result is thus made evident. The synthesis procedure may now be described as follows.

Starting from the expression for $Y_{12}(\lambda)$ as given by Eq. (6), let the denominator polynomial be written in factored form

$$Y_{12}(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n}{b_n(\lambda + \sigma_1)(\lambda + \sigma_2)\cdots(\lambda + \sigma_{2n-1})}, \tag{24}$$

in which

$$0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{2n-1} < \infty. \tag{25}$$

Now form the polynomial

$$q_1(\lambda) = A(\lambda + \sigma_2)(\lambda + \sigma_4) \cdots (\lambda + \sigma_{2n}) \tag{26}$$

with σ-values chosen so that

$$0 < \sigma_1 < \sigma_2 < \sigma_3 < \sigma_4 < \cdots < \sigma_{2n-1} < \sigma_{2n} < \infty, \tag{27}$$

and a value of A such that $q_1(0) < q(0)$, or

$$A\sigma_2\sigma_4 \cdots \sigma_{2n} < b_0 = b_n\sigma_1\sigma_3 \cdots \sigma_{2n-1}. \tag{28}$$

Next form the polynomial $q_2(\lambda) = q(\lambda) - q_1(\lambda)$ and then make the identifications indicated in Eq. (22).

Now consider

$$y_{22} = \frac{q_2(\lambda)}{q_1(\lambda)} = \frac{h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_n\lambda^n}{d_0 + d_1\lambda + d_2\lambda^2 + \cdots + d_n\lambda^n}. \tag{29}$$

Write:

$$\frac{1}{y_{22}} = \frac{d_n\lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_1\lambda + d_0}{h_n\lambda^n + h_{n-1}\lambda^{n-1} + \cdots + h_1\lambda + h_0} = \frac{q_1}{q_2} \tag{30}$$

$$= r_1 + \frac{d'_{n-1}\lambda^{n-1} + d'_{n-2}\lambda^{n-2} + \cdots + d'_1\lambda + d'_0}{h_n\lambda^n + h_{n-1}\lambda^{n-1} + \cdots + h_1\lambda + h_0} = r_1 + \frac{q'_1}{q_2}. \tag{31}$$

The last result is obtained through dividing q_2 into q_1 by the process of long division, which yields the quotient $r_1 = d_n/h_n$ and the remainder q'_1/q_2. Next form:

$$1/y_{22} = r_1 + 1/(q_2/q'_1), \tag{32}$$

and divide q'_1 into q_2, obtaining a quotient $c_1\lambda = h_n\lambda/d'_{n-1}$ and a remainder

$$\frac{q'_2}{q'_1} = \frac{h'_{n-1}\lambda^{n-1} + h'_{n-2}\lambda^{n-2} + \cdots + h'_1\lambda + h'_0}{d'_{n-1}\lambda^{n-1} + d'_{n-2}\lambda^{n-2} + \cdots + d'_1\lambda + d'_0}. \tag{33}$$

The result so far may be written in the form

$$\frac{1}{y_{22}} = r_1 + \cfrac{1}{c_1\lambda + \cfrac{1}{(q'_1/q'_2)}}. \tag{34}$$

The quotient of polynomials q'_1/q'_2 is now treated in the same manner as q_1/q_2, giving,

$$\frac{1}{y_{22}} = r_1 + \cfrac{1}{c_1\lambda + \cfrac{1}{r_2 + \cfrac{1}{c_2\lambda + \cfrac{1}{(q''_1/q''_2)}}}}, \tag{35}$$

whereupon q''_1/q''_2 is again dealt with in the same manner. After n cycles the

remainder is a constant. One thus obtains the following continued fraction expansion

$$\frac{1}{y_{22}} = r_1 + \cfrac{1}{c_1\lambda + \cfrac{1}{r_2 + \cfrac{1}{c_2\lambda + \cdots + \cfrac{1}{r_n + \cfrac{1}{c_n\lambda + \cfrac{1}{r_{n+1}}}}}}}. \tag{36}$$

To this form for y_{22} there corresponds the ladder network shown in Fig. 4. This network has the short-circuit driving-point admittance y_{22} and the short circuit transfer admittance $y_{12}^{(0)} = A_0/q_1$ in which A_0 is a positive real constant. The function $y_{12}^{(0)}$ should be calculated from the network just obtained so as to form a check on the work so far and also to determine the value of the constant A_0.

Now consider again the function y_{22} in the form:

$$y_{22} = \frac{q_2(\lambda)}{q_1(\lambda)} = \frac{h_0 + h_1\lambda + \cdots + h_n\lambda^n}{d_0 + d_1\lambda + \cdots + d_n\lambda^n}. \tag{37}$$

Divide q_1 into q_2 so as to obtain the quotient $g_1 = h_0/d_0$ and the remainder

$$\frac{q_2'}{q_1} = \frac{h_1'\lambda + h_2'\lambda^2 + \cdots + h_n'\lambda^n}{d_0 + d_1\lambda + \cdots + d_n\lambda^n}. \tag{38}$$

Write the result so far

$$y_{22} = g_1 + 1/(q_1/q_2'). \tag{39}$$

Next divide q_2' into q_1 so as to obtain the quotient $s_1\lambda^{-1}$ and the remainder

$$\frac{q_1'}{q_2'} = \frac{d_1'\lambda + d_2'\lambda^2 + \cdots + d_n'\lambda^n}{h_1'\lambda + h_2'\lambda^2 + \cdots + h_n'\lambda^n}. \tag{40}$$

Thus

$$y_{22} = g_1 + \frac{1}{s_1\lambda^{-1} + (q_1'/q_2')}. \tag{41}$$

The same process is not repeated. Instead the function q_1'/q_2' is written

$$\frac{q_1'}{q_2'} = \frac{d_n'\lambda^{n-1} + d_{n-1}'\lambda^{n-2} + \cdots + d_2'\lambda + d_1'}{h_n'\lambda^{n-1} + h_{n-1}'\lambda^{n-2} + \cdots + h_2'\lambda + h_1'} \tag{42}$$

and is manipulated in the same manner as the function $1/y_{22} = q_1/q_2$ in the previous procedure. One thus finally obtains the representation

$$y_{22} = g_1 + \cfrac{1}{s_1\lambda^{-1} + r_2 + \cfrac{1}{c_2\lambda + \cfrac{1}{\ddots + \cfrac{1}{r_n + \cfrac{1}{c_n\lambda + \cfrac{1}{r_{n+1}}}}}}}, \tag{43}$$

to which there corresponds the network of Fig. 5. The values of the r's and c's are, of course, different from those obtained for the previous network. The present network has the short-circuit driving-point admittance y_{22} and the short-circuit transfer admittance $y_{12}^{(1)} = A_1\lambda/q_1$ in which A_1 is a positive real constant. The function $y_{12}^{(1)}$ should be calculated for the network just obtained so as to check the preceding work and to determine the value of the constant A_1.

Now again consider y_{22} as for the process just completed and carry through the same two initial steps, leaving the result in the form

$$y_{22} = g_1 + \frac{1}{s_1\lambda^{-1} + (q_1'/q_2')}. \tag{44}$$

This time the same cycle is repeated a second time so as to yield

$$y_{22} = g_1 + \cfrac{1}{s_1\lambda^{-1} + \cfrac{1}{g_2 + \cfrac{1}{s_2\lambda^{-1} + (q_1''/q_2'')}}}. \tag{45}$$

The remainder at this stage is written

$$\frac{q_1''}{q_2''} = \frac{d_n''\lambda^{n-2} + d_{n-1}''\lambda^{n-3} + \cdots + d_3''\lambda + d_2''}{h_n''\lambda^{n-2} + h_{n-1}''\lambda^{n-3} + \cdots + h_3''\lambda + h_2''} \tag{46}$$

and is manipulated in the same manner as the function q_1'/q_2' in the immediately preceding expansion of y_{22}. Thus one obtains the representation

$$y_{22} = g_1 + \cfrac{1}{s_1\lambda^{-1} + \cfrac{1}{g_2 + \cfrac{1}{s_2\lambda^{-1} + r_3 + \cfrac{1}{c_3\lambda + \cfrac{1}{\ddots + \cfrac{1}{r_n + \cfrac{1}{c_n\lambda + \cfrac{1}{r_{n+1}}}}}}}}}, \tag{47}$$

to which there corresponds the network of Fig. 6. This network also has the same y_{22}-function and a transfer function $y_{12}^{(2)} = A_2\lambda^2/q_1$ in which A_2 is a positive real constant.

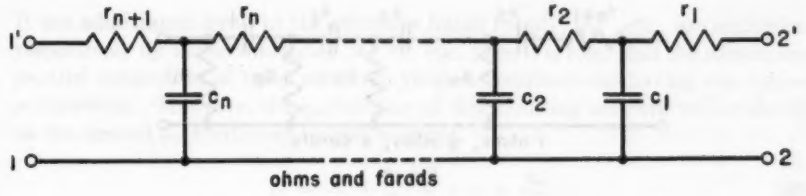


FIG. 4

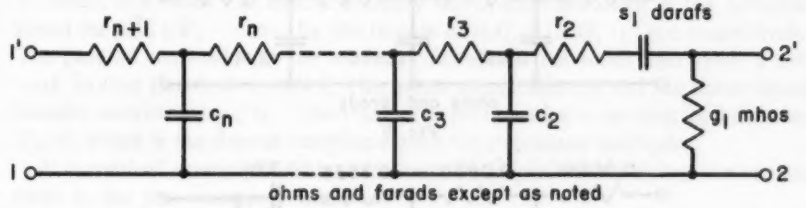


FIG. 5

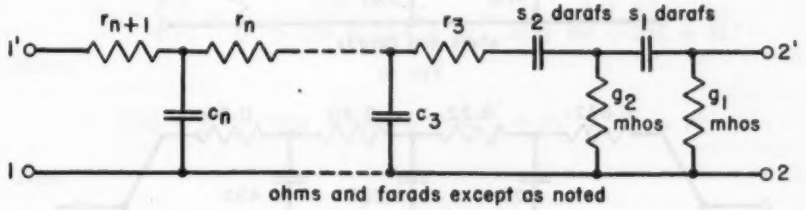


FIG. 6

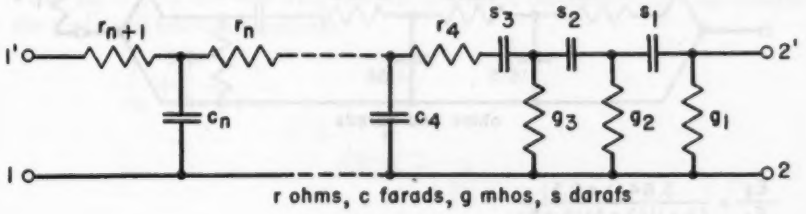


FIG. 7

The next expansion of y_{22} follows the pattern of the preceding one for *three* cycles and then finishes as in the previous cases, thus yielding the network of Fig. 7 which again has the same y_{22}-function but a transfer function $y_{12}^{(3)} = A_3\lambda^3/q_1$ as should be verified by calculation and the constant A_3 determined.

Continuing in this way, one finally obtains a network having the functions y_{22} and $y_{12}^{(n)} = A_n\lambda^n/q_1$. This network has the form shown in Fig. 8. The function $y_{12}^{(n)}$ should be checked and A_n determined.

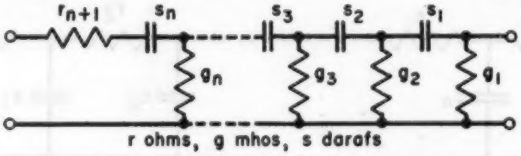


FIG. 8

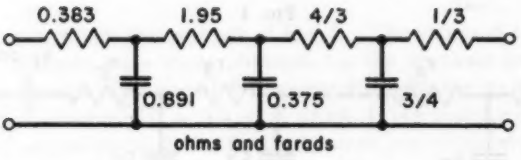


FIG. 9

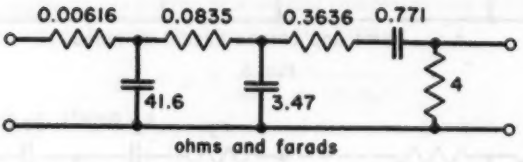
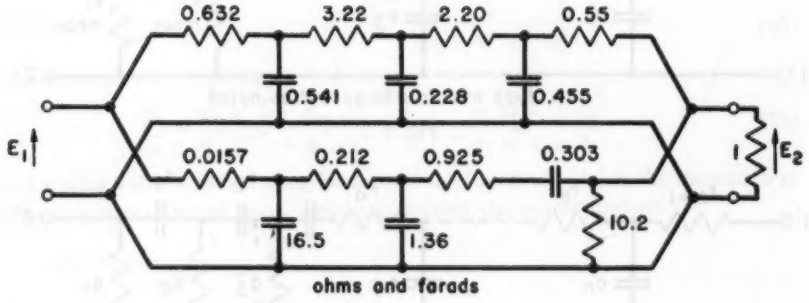


FIG. 10



$$\frac{E_2}{E_1} = \frac{3.64(\lambda+0.5)}{(\lambda+1)(\lambda+3)(\lambda+5)}$$

FIG. 11

Now it should be recalled that the y_{12}-function of the desired network is

$$y_{12} = \frac{p(\lambda)}{q_1(\lambda)} = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n}{q_1(\lambda)}, \tag{48}$$

or

$$y_{12} = \frac{a_0 y_{12}^{(0)}}{A_0} + \frac{a_1 y_{12}^{(1)}}{A_1} + \cdots + \frac{a_n y_{12}^{(n)}}{A_n}. \tag{49}$$

If the admittance levels of the networks found for $y_{12}^{(0)}$, $y_{12}^{(1)}$, etc., are multiplied respectively by the factors a_0/A_0, a_1/A_1, etc., then it is clear that the subsequent parallel connection of these networks yields a resultant one having the desired y_{12}-function. However, the y_{22}-function of this resulting network will evidently be the desired y_{22} multiplied by the constant.

$$G = \frac{a_0}{A_0} + \frac{a_1}{A_1} + \cdots + \frac{a_n}{A_n}. \tag{50}$$

To avoid this result one should multiply the admittance levels of the networks found for $y_{12}^{(0)}$, $y_{12}^{(1)}$, $\cdots$ etc., by the factors a_0/A_0G, a_1/A_1G, $\cdots$ etc. respectively. The parallel connection of the resultant component networks then yields a network having the short-circuit driving-point admittance y_{22} and the short-circuit transfer admittance y_{12}/G. The transfer admittance for a one-ohm load becomes Y_{12}/G, which is the desired function except for a constant multiplier.

A numerical example at this point may be helpful in clarifying the various steps in the procedure just described.

Given:

$$Y_{12} = \frac{p(\lambda)}{q(\lambda)} = \frac{\lambda + 0.5}{(\lambda+1)(\lambda+3)(\lambda+5)} = \frac{\lambda + 0.5}{\lambda^3 + 9\lambda^2 + 23\lambda + 15},$$

choose

$$q_1(\lambda) = \tfrac{1}{4}(\lambda+2)(\lambda+4)(\lambda+6) = \frac{\lambda^3}{4} + 3\lambda^2 + 11\lambda + 12.$$

Then

$$q_2(\lambda) = q(\lambda) - q_1(\lambda) = \tfrac{3}{4}\lambda^3 + 6\lambda^2 + 12\lambda + 3.$$

Using $y_{22} = q_2/q_1$ and following the first procedure described above one obtains the continued fraction expansion

$$\frac{1}{y_{22}} + \frac{1}{3} + \cfrac{1}{\frac{3}{4}\lambda + \cfrac{1}{\frac{1}{3} + \cfrac{1}{0.375\lambda + \cfrac{1}{1.95 + \cfrac{1}{0.891\lambda + \cfrac{1}{0.383}}}}}}$$

and the network shown in Fig. 9. One calculates

$$y_{12}^{(0)} = \frac{3}{\frac{1}{4}\lambda^3 + 3\lambda^2 + 11\lambda + 12}.$$

Using the second procedure described above, one obtains the expansion

$$y_{22} = \frac{1}{4} + \cfrac{1}{1.296\lambda^{-1} + \frac{4}{11} + \cfrac{1}{3.47\lambda + \cfrac{1}{0.0835 + \cfrac{1}{41.6\lambda + \cfrac{1}{0.00616}}}}},$$

and the network shown in Fig. 10. One calculates

$$y_{12}^{(1)} = \frac{9.25\lambda}{\frac{1}{4}\lambda^3 + 3\lambda^2 + 11\lambda + 12}.$$

Here

$$G = \frac{0.5}{3} + \frac{1}{9.25} = 0.275.$$

The admittance levels of the preceding networks should be multiplied by the factors

$$\frac{0.5}{3 \times 0.275} = 0.607, \quad \text{and} \quad \frac{1}{9.25 \times 0.275} = 3.93$$

respectively (divide the resistance values by these factors and multiply the capacitance values by them). Their parallel connection terminated in a 1-ohm load, then has the transfer admittance $1/0.275 = 3.64$ times the Y_{12}-function assumed. The resultant network is shown in Fig. 11.

It may be thought in cases in which n is large, that the computation of the numerical constants A_r in the expressions for $y_{12}^{(r)}$ becomes tedious. So it would if this entire transfer function had to be determined for each corresponding ladder network; but this is not the case as the following discussion shows.

For small values of λ, a typical transfer function assumes the form

$$y_{12}^{(r)} \to A_r\lambda^r/q_1(0), \tag{51}$$

in which $q_1(0)$ is simply the constant term in the polynomial $q_1(\lambda)$. To evaluate A_r it is, therefore, only necessary to determine this limiting form of the transfer function for the appropriate ladder network whose elements at this stage of the procedure are known.

Consider the network of Fig. 7 as a typical one, for which the constant A_3 is to be found. Since for $\lambda \to 0$ the impedances of the shunt capacitances $c_n \cdots c_4$ are very large compared to the resistances r_{n+1}, r_n, $\cdots$ r_4, and the currents through these resistances are negligible because of the series elastance s_3, it becomes clear that the voltage at the input of the ladder network is equal to that just to the left of the elastance s_3. Therefore, only the network to the right of this point influences the value of A_3.

Let the input voltage be denoted by E_1, and those voltages across the conductances g_3 and g_2 by E_3 and E_2 respectively. Since E_1 is the voltage just to the

left of s_3, and the impedance looking to the right of g_3 is very large compared to $1/g_3$, one has

$$\frac{E_3}{E_1} \cong \frac{1/g_3}{(s_3/\lambda) + (1/g_3)} \to \frac{\lambda}{g_3 s_3}. \tag{52}$$

Similarly

$$\frac{E_2}{E_3} \to \frac{\lambda}{g_2 s_2}. \tag{53}$$

The current through the short-circuited output terminals being

$$I_2 = E_2 \lambda / s_1, \tag{54}$$

Eqs. 52 and 53 yield

$$\frac{I_2}{E_1} \to \frac{\lambda^3}{g_2 g_3 s_1 s_2 s_3} \quad \text{for} \quad \lambda \to 0, \tag{55}$$

so that, using Eq. 51, the desired constant becomes

$$A_3 = \frac{q_1(0)}{g_2 g_3 s_1 s_2 s_3}. \tag{56}$$

A modified procedure yielding a network with fewer elements. Consider any of the ladder networks having the transfer admittance

$$y_{12}^{(\nu)} = A_\nu \lambda^\nu / q_1(\lambda) \tag{57}$$

to be represented as shown in Fig. 12 in which only the first two ladder elements are shown explicitly, while the balance of the network is abbreviated by the box B. Since $y_{12} = y_{21}$, one may alternately interpret the transfer admittance as the ratio I_1/E_2 for the network in reversed orientation as illustrated in Fig. 13. Here E denotes whatever voltage rise appears across the capacitance c_n in the arrow direction indicated.

If one recognizes that this voltage is that appearing across the parallel combination of c_n and r_{n+1}, it is clear that an interchange of these two circuit elements has no effect upon E, and that one may redraw the right-hand end of the network as shown in Fig. 14. The short-circuit current at the terminals 1-1′ is, of course, changed to a new value I_1'. By inspection of Figs. 13 and 14 one sees that $I_1 = E/r_{n+1}$ while $I_1' = Ec_n\lambda$, whence

$$I_1'/I_1 = r_{n+1} c_n \lambda. \tag{58}$$

Since $y_{12}^{(\nu)} = I_1/E_2$, one has

$$I_1'/E_2 = y_{12}^{(\nu)} \times r_{n+1} c_n \lambda \tag{59}$$

which, apart from a constant multiplier, is $y_{12}^{(\nu+1)}$.

This result leads one to recognize that a single ladder network is capable of yielding a transfer admittance of the form

$$y_{12}^{(\nu)} \times \left(1 + \frac{a_{\nu+1}}{a_\nu}\lambda\right), \tag{60}$$

thus representing two terms in the expression 23 for y_{12} instead of only one. The appropriate modification in the ladder network is shown in Fig. 15. Here one has

$$I_1'' = E\left(\frac{a}{r_{n+1}} + bc_n\lambda\right) = ay_{12}^{(r)}\left(1 + \frac{br_{n+1}c_n\lambda}{a}\right), \tag{61}$$

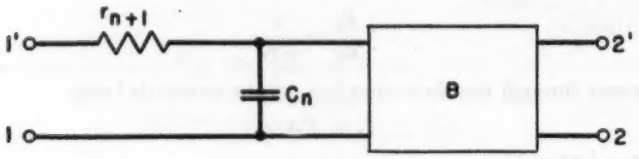


FIG. 12

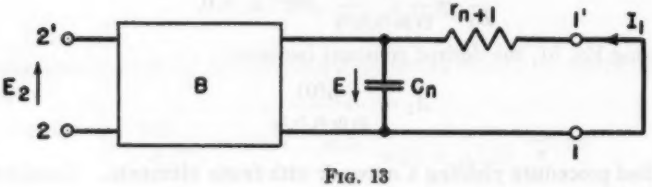


FIG. 13

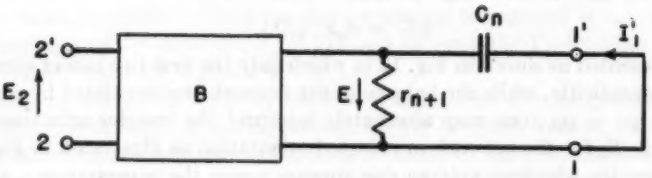


FIG. 14

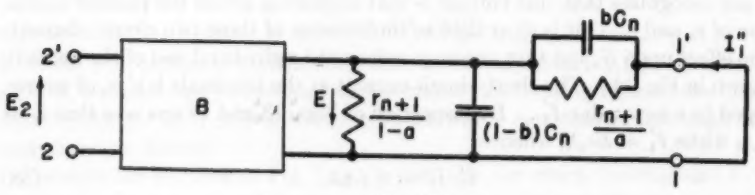


FIG. 15

in which a and b are numerics having values between zero and unity. Comparing Eqs. 60 and 61 it is clear that the desired result is obtained if

$$\frac{br_{n+1}c_n}{a} = \frac{a_{r+1}}{a_r}, \quad \text{or} \quad \frac{a}{b} = \frac{a_r}{a_{r+1}} \times r_{n+1}c_n. \tag{62}$$

The further saving of one resistance or one capacitance element is seen to be possible since a may be chosen equal to unity if $a/b > 1$, or b may be chosen equal

to unity if $a/b < 1$. Thus only two c's and one r, or two r's and one c are needed in the terminating branches. One may say that through the use of one additional r or one additional c, any one of the component ladder networks may be modified at its input end so as to serve the purpose of two ladder networks in the resultant configuration.

As an illustration of this modified procedure, consider the numerical example given above. Since only two ladders are involved, it should be possible to obtain the desired result through a modification of the component ladder of Fig. 9 (having the transfer admittance $y_{12}^{(0)}$) alone. Here

$$r_{n+1}c_n = 0.383 \times 0.891 = 0.341,$$

and since in this problem $a_1/a_0 = 2$, Eq. 62 yields

$$\frac{a}{b} = \tfrac{1}{2} \times 0.341 = 0.1705.$$

Since this number is less than unity, one may choose $b = 1$, whence $a = 0.1705$. The shunt capacitance $(1 - b)c_n$ in Fig. 15 drops out; the shunt resistance

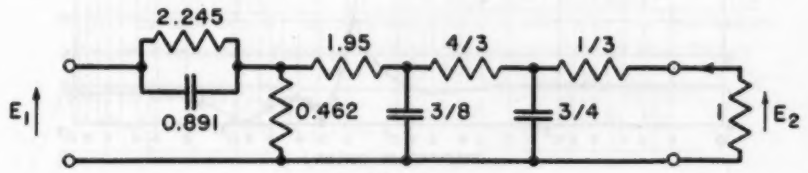


FIG. 16

$r_{n+1}/(1 - a)$ has the value $0.383/0.8295 = 0.462$; the capacitance and resistance in the parallel combination appearing in the final series branch are $bc_n = c_n = 0.891$, and $r_{n+1}/a = 0.383/0.1705 = 2.245$.

Noting that in Fig. 15 the network is turned end-for-end, one thus recognizes that the desired solution is given by the network shown in Fig. 16. One readily finds that here $E_2/E_1 = 1.02$ times the stated Y_{12}-function, yielding a somewhat smaller gain than the solution given in Fig. 11. It is not appropriate at this stage, however, to jump at the conclusion that the alternate procedure which realizes the desired result with fewer elements is necessarily always inferior from a gain standpoint.

With regard to the question of gain it should be pointed out that in the design of filter networks required to have significant rates of discrimination in the vicinity of cutoff, one finds rather high values of constant loss, a circumstance suggesting that it is more appropriate to realize such a design through several stages with intervening vacuum tubes providing gain. Decomposition into stages, while in some cases absolutely necessary to prevent the signal from being attentuated below the noise level, turns out to be effective in reducing computational work and also the over-all loss, which is usually smaller than it would be if the entire characteristic were obtained with a single complex network.

The total number of elements involved in a design. Suppose first that the synthesis is carried out according to the method which does not combine two ladders into one but yields the result in the form shown in Fig. 11 for the numerical example given above.

If the trigonometric polynomial 11 used in the approximation problem termi-

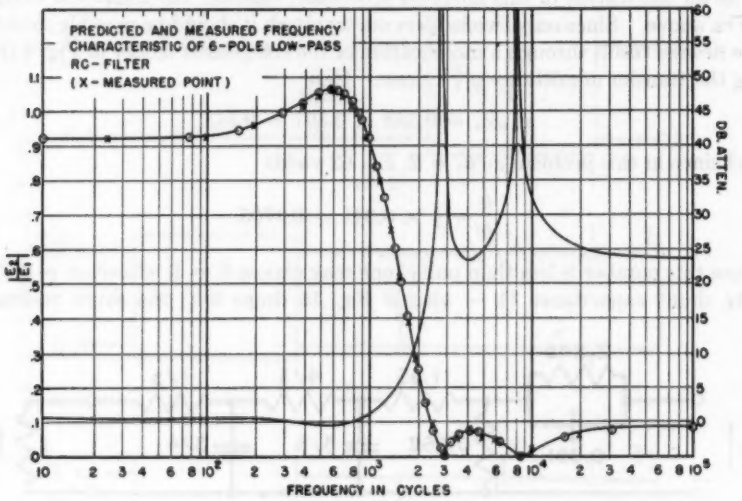

PREDICTED AND MEASURED FREQUENCY CHARACTERISTIC OF 6-POLE LOW-PASS RC-FILTER
(X-MEASURED POINT)
DB ATTEN.
FREQUENCY IN CYCLES

FIG. 17

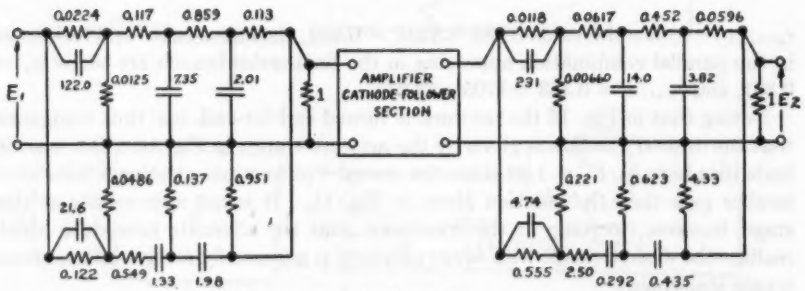


FIG. 18. NORMALIZED SIX POLE, LOW PASS RC-FILTER VALUES IN OHMS AND FARADS

nates with cos $n\phi$, the preceding discussion shows that a total of $n + 1$ ladder networks are needed, of which each requires $n + 1$ resistances and n capacitances. One thus obtains a total of $(n + 1)^2$ resistances and $n(n + 1)$ capacitances, or altogether $(n + 1)(2n + 1)$ elements.

If the procedure which combines pairs of ladder networks into one is followed, best economy is achieved when the total number of ladders according to the first procedure is even—hence for n odd. Each modified ladder then has *either* $(n +$

1) r's and $(n + 1)$ c's, *or* $(n + 2)$ r's and n c's. Since there are $(n + 1)/2$ such modified ladders in parallel, one has a total of r's between $(n + 1)^2/2$ and $(n + 1)$

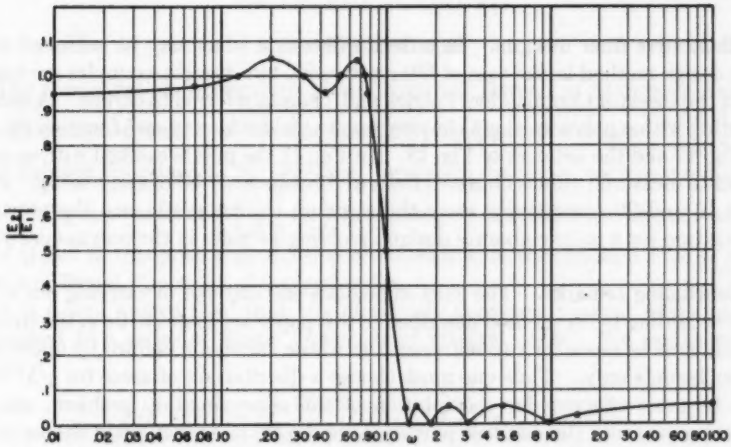


FIG. 19. NORMALIZED FREQUENCY CHARACTERISTIC FOR THE 14 POLE FILTER

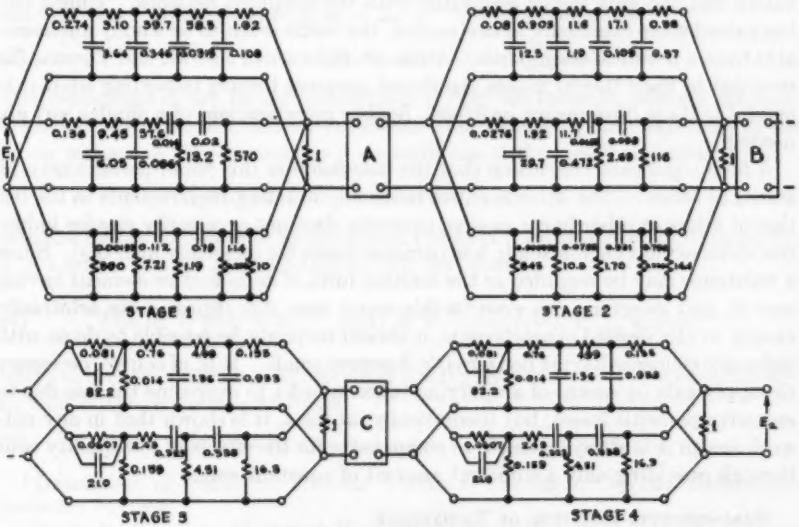


FIG. 20. VALUES IN OHMS AND FARADS
A, B, C—Amplifier, cathode-follower
Normalized 14 pole filter

$(n + 2)/2$, and a total of c's between $(n + 1)^2/2$ and $n(n + 1)/2$, yielding altogether a grand total of $(n + 1)^2$ elements of which half or less are capacitances.

When n is large, the first procedure yields approximately $2n^2$ elements and the modified one n^2 elements, of which roughly half are capacitances and half are resistances.

Illustrative filter designs. In order to illustrate what may be achieved with this design method in the case of filter networks, two specific examples are taken from the thesis of Commanders Patrick and Thomas referred to earlier. A design starting with a polynomial $q(\lambda)$ having 6 poles yields the response function shown in Fig. 17 and the network of Fig. 18. In Fig. 17 the points marked with crosses indicate measured values obtained from a "breadboard" laboratory setup. Figures 19 and 20 respectively show the response characteristic and the network realization for a more elaborate design involving 14 poles in the polynomial $q(\lambda)$.

Concluding remarks. The chief difficulties encountered in carrying out a design according to the method described in this paper are found to lie in the choices which must be made for various quantities whose values are defined by upper and lower bounds only. Thus one must choose a distribution of zeros for $q(\lambda)$ that will be most effective for the solution of the approximation problem, and in splitting $q(\lambda)$ into the additive parts $q_1(\lambda)$ and $q_2(\lambda)$ one must again choose locations of zeros within given limits. The choices made, affect the spread of element values and the gain factor associated with the resultant network. Unless one has gained some experience in this matter, the result is apt to be wholly unreasonable from a practical standpoint. Although the work of Patrick and Thomas (as recorded in their thesis) makes significant progress toward indicating what it is helpful to do in filter design problems, further investigations of a similar sort are needed.

A final significant conclusion that the discussion in this paper permits may be stated as follows: The attainment of unusually exacting requirements in the design of filters or other linear passive networks does not necessarily require inductive elements having unusually low parasitic losses (or unusually high Q's). Since a resistance may be regarded as the limiting form of an inductive element having zero Q, and since one can even in this worst case still approximate arbitrarily closely to any desired characteristic, it should certainly be possible to do so with inductive elements having nónzero Q's, however small. It is, of course, necessary to supply gain by means of amplifying tubes in order to overcome the loss due to excessive parasitic losses; but theoretically, at least, it is shown that in any network design it is always possible to compensate for the effects of low-quality coils through providing only a sufficient amount of constant gain.

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(Received September 27, 1948)

NOTE ON THE EXPANSION OF CONFLUENT HYPERGEOMETRIC FUNCTIONS IN TERMS OF BESSEL FUNCTIONS OF INTEGRAL ORDER

By Meyer Karlin

The Computation Laboratory of the National Bureau of Standards has recently published Tables of the Confluent Hypergeometric Function $F\left(\frac{n}{2}, \frac{1}{2}; x\right)$ and Related Functions.[1] Mr. William Horenstein developed various methods of evaluating the function $F\left(\frac{n}{2}, \frac{1}{2}; x\right)$, depending on the relative values of n and x. The object of this note is to develop an expansion of the function $F(\alpha, \gamma; x)$ in terms of Bessel Functions of integral order.

It is a well known fact[2] that any absolutely convergent power series can be expressed as a series of Bessel Functions of integral order. We can therefore assume[3]

$$F(\alpha, \gamma; x) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha + k)}{\Gamma(\alpha)\Gamma(\gamma + k)} \cdot \frac{x^k}{k!} = \sum_{k=0}^{\infty} c_k I_k(x) \tag{1}$$

The boundary conditions $F(\alpha, \gamma; 0) = 1$ and $\{(d/dx)F(\alpha, \gamma; x)\}_{x=0} = \alpha/\gamma$ yield the first two coefficients $c_0 = 1$, $c_1 = 2\alpha/\gamma$. We can thus write

$$F(\alpha, \gamma; x) = I_0(x) + 2\alpha/\gamma I_1(x) + \sum_{k=2}^{\infty} c_k I_k(x) \tag{2}$$

The coefficients c_k can be calculated from (1) by the formula, relating the coefficients in the expansion in terms of Bessel Functions with the coefficients in the power series. A simpler method is to make use of the recurrence relation

$$(d/dx)F(\alpha, \gamma; x) = (\alpha/\gamma)F(\alpha + 1, \gamma + 1; x) \tag{3}$$

satisfied by the confluent hypergeometric function. Substituting in formula (2), $\alpha + 1$ and $\gamma + 1$ for α and γ respectively, we obtain

$$F(\alpha + 1, \gamma + 1; x) = I_0(x) + 2\frac{\alpha + 1}{\gamma + 1} I_1(x) + \sum_{k=2}^{\infty} d_k I_k(x) \tag{4}$$

where d_k is obtained from c_k by replacing α and γ by $\alpha + 1$ and $\gamma + 1$ respectively. In view of the recurrence relations

$$2I_k'(x) = I_{k-1}(x) + I_{k+1}(x), \qquad I_0'(x) = I_1(x) \tag{5}$$

[1] Department of Commerce, National Bureau of Standards. Applied Mathematics Series No. 3, (1949).

[2] See for instance Gray, Mathews & MacRobert's "Treatise on Bessel Functions and their application to Physics," London, 1931, p. 34.

[3] We chose the functions $I_k(x)$ rather than $J_k(x)$, because the functions $I_k(x)$ and $F(\alpha, \gamma; x)$ increase without limit as $x \to \infty$ while the functions $J_k(x)$ oscillate between -1 and $+1$.

satisfied by the Bessel Functions $I_k(x)$, equation (3) becomes

$$\frac{\alpha}{\gamma} I_0(x) + I_1(x) + \frac{\alpha}{\gamma} I_2(x) + \frac{1}{2}\sum_{k=3}^{\infty} c_k[I_{k-1}(x) + I_{k+1}(x)]$$

$$= \frac{\alpha}{\gamma} I_0(x) + 2\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} I_1(x) + \frac{\alpha}{\gamma}\sum_{k=2}^{\infty} d_k I_k(x) \tag{6}$$

Equating coefficients of $I_k(x)$ $(k = 1, 2, \cdots)$ on both sides of the equation we get the following recurrence relations

$$\begin{cases} \frac{1}{2}c_2 + 1 = 2\dfrac{\alpha(\alpha+1)}{\gamma(\gamma+1)}; \\ \dfrac{\alpha}{\gamma} + \frac{1}{2}c_3 = \dfrac{\alpha}{\gamma} d_2; \quad \text{where} \quad d_2 = 4\dfrac{(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)} - 2 \\ \frac{1}{2}(c_{k+1} + c_{k-1}) = \dfrac{\alpha}{\gamma} d_k \qquad k \geq 3 \end{cases} \tag{7}$$

As an illustration of the above, let $\alpha = 41/2, \gamma = \frac{1}{2}$ then we obtain

$$c_0 = 1, \quad c_1 = 82, \quad c_2 = \frac{7046}{3}, \quad c_3 = 42066, \quad c_4 = \frac{11734570}{21},$$

$$c_5 = \frac{17927414}{3}, \quad c_6 = \frac{4156610842}{77}, \quad c_7 = \frac{66784838795}{143} \text{ etc.},$$

and therefore

$$F\left(\frac{41}{2}, \frac{1}{2}; x\right) = I_0(x) + 82I_1(x) + \frac{7046}{3} I_2(x) + 42066I_3(x)$$

$$+ \frac{11734570}{21} I_4(x) + \frac{17927414}{3} I_5(x) + \frac{4156610842}{77} I_6(x)$$

$$+ \frac{66784838795}{143} I_7(x) + \cdots \tag{8}$$

Eight terms in this expansion yield $F(20.5, \frac{1}{2}; .2) = 30.97745$ as compared with 30.97819 obtained from the tables mentioned above and computed by a method different than that here described. To obtain a comparable accuracy from the power series, hundreds of terms would be needed.

THE COMPUTATION LABORATORY
NATIONAL BUREAU OF STANDARDS

(Received June 20, 1948)

ON THE LONG-RUN ERROR IN THE NUMERICAL SOLUTION OF CERTAIN DIFFERENTIAL EQUATIONS

By Warren S. Loud

1. Introduction. The purpose of this paper is to examine numerical solutions of differential equations from the point of view of their long-run accuracy. The main result is an estimate for the long-run relative error due to truncation in the case of linear differential equations with constant coefficients. There is defined an index which can be used to measure long-run accuracy of a numerical process. Finally a construction of a class of numerical methods suited to long-run solutions is given.

There are three reasons for the restriction to linear equations with constant coefficients. First there are no difficulties with long-run existence of solutions. Second, linear equations with constant coefficients can be solved explicitly, and the numerical solution can also be found explicitly. This furnishes a firm basis for comparison of the numerical solution with the true solution. Third, if a numerical method is to be of value in solving general differential equations, it should be extremely reliable with the simplest types, and should be easily controlled for them.

2. Methods Considered. A numerical solution of a differential equation

$$dy/dx = f(x, y) \tag{2.1}$$

will produce a sequence of values of y corresponding to a given sequence of values of x. For our purposes we assume that the values of x are equally spaced, with separation h, and that the initial value of x is zero. We shall use the notation

$$x_n = nh \qquad y_n = y(x_n) = y(nh) \qquad y'_n = y'(nh). \tag{2.2}$$

The numerical methods studied are all formulae for approximations to y_{n+1} in terms of the coordinates of a finite number of the points (x_n, y_n), (x_{n-1}, y_{n-1}), $\cdots$. The formulae used can all be derived from polynomial approximations of the functions $y(x)$ and $y'(x)$, or equivalently by using a portion of the Taylor series for $y(x)$ and $y'(x)$ in which the derivatives are replaced by their finite-difference approximations, correct to the order of magnitude of the first neglected term.

Typical of such methods found in the literature are the Runge-Kutta method, Milne's method, and a class of methods described on pages 225 ff of Scarborough[1]. In a later section of this paper an additional class of methods of this same type is described.

3. Numerical Solution of Linear Differential Equations with Constant Coefficients. In the case of linear differential equations with constant coefficients

[1] See the bibliography at the end.

the formulae associated with numerical methods of the type described in the preceding section become linear difference equations with constant coefficients, which can be solved explicitly.

We restrict ourselves to equations in which the independent variable does not appear explicitly. Any such equation, or system of such equations can be reduced to a system of simultaneous first-order equations of the same type. By use of matrices the system can be written

$$Y'(x) = A\ Y(x) \tag{3.1}$$

where Y is a column matrix and A is a constant square matrix. The solution of (3.1) is

$$Y(x) = e^{Ax}\ Y_0 \tag{3.2}$$

where Y_0 is the column matrix of initial values, and e^{Ax} is the conventional exponential function of the matrix Ax. Thus the true value of $Y(nh) = Y_n$ is given by

$$Y_n = e^{Anh}\ Y_0. \tag{3.3}$$

When (3.1) is solved by a numerical process of the type under consideration, the solutions of the difference equations have the form

$$Y_n = \varphi_1^n\ C_1 + \cdots + \varphi_k^n\ C_k \tag{3.4}$$

where the φ's are matrix functions of the matrix Ah, the C's are constant column matrices, and k is equal to the number of previous values of y used in the formula for y_{n+1}. One of the φ's is an approximation to e^{Ah}, while the others all tend to zero as h tends to zero.

The matrix solutions can be interpreted as systems of solutions. If there are m equations in the system (3.1), (3.3) can be interpreted as the system

$$y_j(nh) = \sum_{k=1}^{m} c_{jk} e^{\lambda_k nh} \qquad j = 1, 2, \cdots, m. \tag{3.5}$$

The λ's are the characteristic values of the matrix A. The form of (3.5) is modifield in the usual way if the λ's are not all distinct.

In a similar way (3.4) can be interpreted as a system of solutions. It is found that each term of (3.5) of the form $ce^{\lambda nh}$ is replaced by k terms

$$b_1 m_1^n + \cdots + b_k m_k^n. \tag{3.6}$$

The b's are constant, and the m's are functions of the quantity λh. Moreover they are precisely the same functions of λh that the φ's of (3.4) are of the matrix Ah. This is a consequence of the fact that the characteristic values of an analytic function of a matrix A are related to the characteristic values of A by the same functional relationship.

From the preceding discussion we see that the numerical method will approximate each term of the true solution in the same manner. Therefore it will be sufficient to study the single equation

$$dy/dx = ay \tag{3.7}$$

where a is a constant and y is a single variable. The results can then be applied to the system (3.1).

4. The Long-Run Error. The approximate solution of (3.7) will have the form

$$y_n = b_1 m_1^n + \cdots + b_k m_k^n \tag{4.1}$$

where the b's are constants, and the m's are functions of the quantity ah. One of them, m_1 say, will be an approximation to e^{ah}, while the others will tend to zero with h. In the case of Milne's method there is an exception to the above in that there is a second term which does not tend to zero, and high powers of it may well become large. For this reason this method is not suitable for long-run automatic computation.

The numerical solution of (3.7) will be assumed then to have the form of (3.6) where for large values of n only one term is not negligible. In this term, the quantity e^{nah} will be approximated by the nth power of an expression of the form

$$\begin{aligned} f_r = 1 + ah + (ah)^2/2! + \cdots + (ah)^r/r! \\ + [1/(r+1)! - K_r]\,(ah)^{r+1} + \cdots \end{aligned} \tag{4.2}$$

where K_r is different from zero, and the terms omitted are all of degree $(r+2)$ or higher.

We can now state that if E is the relative error due to replacing e^{ah} by f_r, then for all practical cases

$$|E| < |4nK_r(ah)^{r+1}| \tag{4.3}$$

where r and K_r are defined by (4.2) and n is the number of steps taken.

To show this, we note that the relative error E after n steps is given by

$$E = (f_r^n - e^{nah})/e^{nah} = (f_r e^{-ah})^n - 1. \tag{4.4}$$

Since E is a relative error, it must be less than one in magnitude for practical purposes. If this is so,

$$|E| < 2n\,|\log (f_r e^{-ah})|. \tag{4.5}$$

If the logarithm is expanded into a series we find that

$$\log (f_r e^{-ah}) = K_r(ah)^{r+1} + O(ah)^{r+2}) \tag{4.6}$$

so that if (ah) is sufficiently small, as it will be for practical cases,

$$|\log (f_r e^{-ah})| < |2K_r(ah)^{r+1}|. \tag{4.7}$$

Combining (4.5) with (4.7), the result (4.3) follows.

Using (4.3) we can give a partial answer to the question as to the size of h if the relative error is to remain less than a preassigned quantity over a preassigned range of x. If the maximum of E is to be ϵ over the range $0 \leq x \leq X$, by using (4.3) with $n = X/h$, we find

$$h < \left[\frac{\epsilon}{4X\,|K_r|\,|a|^{r+1}}\right]^{1/r}. \tag{4.8}$$

From the form of the expression (4.8) it is evident that of the two quantities which arise from the choice of numerical method, r is of much more significance than K_r. So far as long-run truncation error is concerned, a method with a high value of r is to be preferred.

5. Other Errors. It was stated earlier that (4.8) gives a partial answer to the question of the size of h. The errors discussed in the foregoing have been so-called truncation errors, and are due to the repeated truncation of the Taylor series for the unknown function. It might appear from what has been said that arbitrarily high accuracy could be obtained by merely making the interval h small enough. This is not the case because of another important source of error called round-off error. At each stage of a numerical computation the numbers used must be rounded off to a finite decimal expansion. The errors so introduced will increase as the number of steps taken increases. This will bring about an effective minimum value of h for a given accuracy. No attempt has been made in this paper to analyze round-off error.

For automatic computation, where large numbers of steps are taken without inspection of intermediate results, it appears that much is to be gained by the use of methods with higher values of r. If N represents the number of steps taken for solution over a fixed range of x, it follows from (4.3) that truncation error is proportional to N^{-r}, so that for a given truncation error maximum a smaller number of steps will be required with the larger value of r, which will reduce the round-off error, which is known to vary approximately as $N^{3/2}$.

For the sake of completeness another source of error should be mentioned. An error will arise from the fact that b_1, the coefficient of m_1 in (4.1) will not be equal to y_0 in general. This error is constant, and if the first few values of y are accurate, will be small.

6. Construction of a Class of Numerical Methods. For each of the methods mentioned earlier the index r can be computed directly. The Runge-Kutta method and Milne's method each have $r = 4$. The methods in Scarborough, which involve polynomial extrapolation of the derivative have $r = 2$, 3, and 4 respectively for polynomials of degrees 1, 2, and 3. The class of methods to be constructed is not essentially different from the above methods. Its principal feature is that it utilizes the values of both y and y' in forming the polynomial approximations. As a result, for a given value of r, fewer previous values of the solution are needed.

The construction is as follows. A polynomial of degree $2k$ or $2k + 1$ is passed through the $k + 1$ points $(x_n, y_n), \cdots (x_{n-k}, y_{n-k})$ having the slope y'_j at the point (x_j, y_j) for all $k + 1$ points if the degree is odd, and for all but (x_{n-k}, y_{n-k}) if the degree is even. The value of this polynomial for x_{n+1} is called y^*_{n+1}. Next the function $y'(x)$ is approximated by a polynomial of degree $k + 1$ passing through the $k + 2$ points $(x_{n-k}, y'_{n-k}), \cdots, (x_n, y'_n), (x_{n+1}, y'^*_{n+1})$ where $y'^*_{n+1} = f(x_{n+1}, y^*_{n+1})$. This last polynomial is then integrated from x_n to x_{n+1}, and the result is used as the approximation to the increment in y from y_n to y_{n+1}.

The value of the index r depends on the degrees of the polynomials used in the above construction. The values of r found when the degree of the first polynomial is one, two, three, four, and five are two, three, three, four, and four respectively.

7. Example. To illustrate the procedures described in the foregoing we give the explicit formula corresponding to a numerical method, and determine the quantities r and K_r. We derive the formula by the method described in the preceding section, using initially a polynomial of degree four. With the notation we have been using, the polynomial of degree four passing through the three points (x_n, y_n), (x_{n-1}, y_{n-1}), and (x_{n-2}, y_{n-2}) having at the first two of these the slopes y'_n and y'_{n-1} respectively, will assume for $x = x_{n+1}$, the value

$$y^*_{n+1} = y_{n-2} + 9y_{n-1} - 9y_n + 6hy'_{n-1} + 6hy'_n. \tag{7.1}$$

We next integrate the polynomial of degree three passing through the four points (x_{n-2}, y'_{n-2}), (x_{n-1}, y'_{n-1}), (x_n, y'_n), and (x_{n+1}, y'^*_{n+1}) between x_n and x_{n+1} to give a formula for y_{n+1}

$$y_{n+1} = y_n + h(y'_{n-2} - 5y'_{n-1} + 19y'_n + 9y'^*_{n+1})/24. \tag{7.2}$$

When the formula (7.2) is applied to the equation (3.7) we obtain

$$\begin{aligned} y_{n+1} = {} & (12 - 31ah + 27a^2h^2)y_n/12 + (38ah + 27a^2h^2)y_{n-1}/12 \\ & + (5ah)y_{n-2}/12. \end{aligned} \tag{7.3}$$

If the values of y_n, y_{n-1}, and y_{n-2} are assumed correct, by direct substitution it is found that y_{n+1} has an error of the order of $(ah)^5$, and that this is still the order of error if y_n, y_{n-1} and y_{n-2} are correct only to that order.

If we now solve the difference equation (7.3) we find

$$\begin{aligned} y_n = {} & c_1[1 + ah + \tfrac{1}{2}(ah)^2 + \tfrac{1}{6}(ah)^3 + \tfrac{1}{24}(ah)^4 + \tfrac{5}{144}(ah)^5 + \cdots]^n \\ & + c_2[(5ah/12)^{\frac{1}{2}} + \cdots]^n + c_3[-(5ah/12)^{\frac{1}{2}} + \cdots]^n. \end{aligned} \tag{7.4}$$

If the first few values of y are nearly correct, the constants c_2 and c_3 will be nearly zero, while c_1 will be close to y_0. It is evident from (7.4) that the second and third terms will become negligible as n increases if (ah) is of practical size. Also comparing with (4.2) it is clear that for this numerical process the index $r = 4$, with $K_r = -19/720$, so that using (4.3) we have

$$|E| < 19n\,(ah)^5/180. \tag{7.5}$$

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(Received August 2, 1948)

ON THE UNIQUENESS OF FLOWS WITH GIVEN STREAMLINES

By R. C. Prim

1. Introduction. In a recent paper[1], D. Gilbarg has posed and partially answered the following question:

Given a steady flow of an ideal fluid (inviscid and with uniformly constant density) in the absence of external force fields, what flows of the same kind exist with the same streamline pattern as the given flow?

Gilbarg's investigation of this question dealt with the case of plane flow patterns and yielded the following results:

(1) If the given flow does not have a constant velocity magnitude along each individual streamline, the only flows with the same streamline pattern are those having velocity fields that are simply proportional to that of the given field.

(2) If the given flow has a constant velocity magnitude along each individual streamline, all other flows having this property are possible flows for the same streamline pattern.

(3) The only *irrotational* flows having a constant velocity magnitude along each streamline have a streamline pattern consisting of concentric circles or (as a limiting case) of parallel straight lines.

In the present paper, the above question is answered for the case of three-dimensional flow patterns, and the geometric implications of the results in the case of general irrotational flows and of rotational plane and axially-symmetric flows are clarified.

2. Analysis of the Basic Question. The principles of conservation of mass and conservation of momentum lead to the following equations governing steady flow of ideal fluids in the absence of body forces:

$$\operatorname{div} \rho \mathbf{v} = 0 \tag{1}$$

and

$$\mathbf{v} \cdot \operatorname{grad} \mathbf{v} + (1/\rho) \operatorname{grad} p = 0 \tag{2}$$

where $\mathbf{v}$ denotes the velocity vector, p the pressure, and ρ the density. These equations yield for the case where ρ is a constant throughout the following necessary and sufficient conditions on $\mathbf{v}$ alone:

$$\operatorname{div} \mathbf{v} = 0, \quad (3) \qquad \operatorname{curl} (\mathbf{v} \cdot \operatorname{grad} \mathbf{v}) = 0 \tag{4}$$

We now formulate our problem as follows: Given a velocity field $\mathbf{v}$ satisfying (3) and (4), under what conditions will the velocity field

$$\mathbf{v}^* = f(x, y, z)\mathbf{v}$$

satisfy the same equations?

[1] D. Gilbarg, *On the Flow Patterns Common to Certain Classes of Plane Fluid Motions*, Jour. of Math. and Phys., **26**, pp. 137–142 (1947).

Equation (3) requires that

$$\operatorname{div} f\mathbf{v} = f \operatorname{div} \mathbf{v} + \mathbf{v}\cdot\operatorname{grad} f = 0$$

or, using (3), that

$$\boxed{\mathbf{v}\cdot\operatorname{grad} f = 0} \tag{5}$$

The condition (4) leads to

$$\operatorname{curl} (f\mathbf{v}\cdot\operatorname{grad} f\mathbf{v}) = f^2 \operatorname{curl} (\mathbf{v}\cdot\operatorname{grad} \mathbf{v}) + \operatorname{grad} f^2 \times (\mathbf{v}\cdot\operatorname{grad} \mathbf{v}) + \operatorname{curl} [f\mathbf{v}(\mathbf{v}\cdot\operatorname{grad} f)] = 0$$

whence, using (5) and (4)

$$\operatorname{grad} f \times (\mathbf{v}\cdot\operatorname{grad} \mathbf{v}) = 0 \tag{6}$$

From (2) it is seen that (6) implies

$$\boxed{\operatorname{grad} f \times \operatorname{grad} p = 0} \tag{7}$$

The condition (7) permits three possible cases:

(a) $grad\ f = 0$. For this relatively trivial case ($\mathbf{v}^*$ simply proportional to $\mathbf{v}$), no restriction of the given flow is required by (5) and f may be assigned any constant value.

(b) $\operatorname{grad} f \neq 0$, $\operatorname{grad} p = 0$. If the given flow is a flow of uniform pressure—and hence necessarily a flow with straight streamlines and with constant velocity magnitude along each individual streamline—(5) only requires that f be constant along each particular streamline.

(c) $\operatorname{grad} f \neq 0$, $\operatorname{grad} p \neq 0$, $f = f(p)$. For this case, (5) can be satisfied if and only if the given flow has a constant pressure—and hence a constant velocity magnitude—along each individual streamline. If the given flow does have this property, the function $f(p)$ is substantially arbitrary (subject, of course, to appropriate differentiability restrictions).

3. Geometric Implications of the Results. In the following we shall call a given flow *unique* when the only flows with the same streamline pattern as that flow are flows having velocity fields that are simply proportional to that of the given flow. The theorems stated are valid for steady flows of an ideal incompressible fluid in the absence of external force fields.

The analysis of Section 2 establishes the following general theorem:

THEOREM 1: *Any flow is unique unless it has a constant velocity magnitude along each individual streamline.*

For *general spatial flows*, the geometric implications of Theorem 1 are unknown. That is, no exhaustive study has been made of general spatial flow fields with constant velocity magnitude along each individual streamline.

However, for the case of *irrotational flow fields* Hamel[2] has proved that the only

[2] G. Hamel, *Potentialstroemungen mit konstanter Geschwindigkeit*, Sitzungsberichte der Preussischen Akademie der Wissenschaft, 5-20 (Berlin 1937).

flows with this property are those helicoidal flows obtainable by normal superposition of a uniform parallel flow and a potential vortex. Hence we have the following corollary:

COROLLARY 1A: *All irrotational flows are unique except those irrotational helicoidal flows obtained by normal superposition of a uniform parallel flow and a potential vortex.*

(It will be noted that this reduces to Gilbarg's result for the case of plane irrotational flow.)

Again for the special case of *plane flow fields* the geometric implications of Theorem 1 are known. Nemenyi and Prim[3] have shown that the only steady plane flows of an ideal gas (rotational or not) in the absence of body forces for which the velocity magnitude remains constant along each individual streamline are those flows having streamline patterns consisting of concentric circles or (as a limiting case) of parallel straight lines. The proof used is valid *a fortiori* for an ideal liquid. Therefore we have:

COROLLARY 1B: *All plane flows are unique except those having concentric circles (or parallel straight lines) as streamlines.*

Finally, for the case of *axially-symmetric flow fields* it will be proved in Section 4 that the only flows with constant velocity magnitude along each streamline are those trivial flows having all streamlines parallel to the axis. This establishes:

COROLLARY 1C: *All axially-symmetric flows are unique except purely axial flows.*

4. Appendix: *Axially-Symmetric Fluid Flow with Constant Velocity Magnitude Along Individual Streamlines.* In order to establish Corollary 1C we investigate the possibility of steady axially-symmetric fluid flows (in the absence of viscosity and external force fields) for which the velocity magnitude is constant along each individual streamline. Although our immediate interest is in flows of ideal incompressible fluids our proof will be valid also for any fluid for which constancy of velocity along a streamline implies constancy of pressure and density along the streamline (e.g., thermally non-conducting compressible fluids).

Letting r and x denote respectively the radial and axial coordinate distance, we introduce the stream function ψ through the definition

$$d\psi/dn = 2\pi r\rho v \tag{8}$$

where dn denotes the differential outward normal to the streamlines. We now consider flows for which $v = v(\psi)$, $\rho = \rho(\psi)$, and $p = p(\psi)$. For these flows the significant part of the dynamic equation (2) can be written

$$dp/dn = \kappa\rho v^2 \tag{9}$$

where $\kappa = \kappa(x, \psi)$ denotes the curvature of the streamlines (taken as positive when the streamlines are concave with respect to the axis). Equations (8) and (9) then imply

$$\kappa(x, \psi) = \frac{2\pi dp(\psi)/d\psi}{v(\psi)}\, r(x, \psi) \equiv A(\psi)r(x, \psi) \tag{10}$$

[3] P. Nemenyi and R. Prim, *Some Properties of Plane Rotational Gas Flows*, Jour. of Math. and Phys., **27**, pp. 130–135 (1948).

wherein $A(\psi)$ is defined. We now introduce the familiar expression for κ

$$\kappa(x, \psi) = \frac{-(\partial q/\partial x)_\psi}{(1 + q^2)^{3/2}} \tag{11}$$

where q is defined by

$$q(x, \psi) \equiv (\partial r/\partial x)_\psi \tag{12}$$

and obtain the equation

$$q(\partial q/\partial r)_\psi + A(\psi)(1 + q^2)^{3/2} r = 0 \tag{13}$$

which yields upon integration

$$\boxed{\sqrt{q^2 + 1} = 2/[A(\psi)r^2 + B(\psi)]} \tag{14}$$

Returning to the streamfunction equation (8), we observe that it may be rewritten in the form

$$d\psi = 2\pi\rho vr[(q^2 + 1)^{-1/2} dr - q(q^2 + 1)^{-1/2} dx]$$

or

$$dr(x, \psi) = q\, dx + [(q^2 + 1)^{1/2}/2\pi\rho vr]\, d\psi \tag{15}$$

The condition of integrability for the function $r(x, \psi)$ now leads to the equation

$$2\pi\rho v \left(\frac{\partial q}{\partial \psi}\right)_r - \frac{q^2}{r\sqrt{q^2 + 1}} \left(\frac{\partial q}{\partial r}\right)_\psi + \frac{q\sqrt{q^2 + 1}}{r^2} = 0$$

which by virtue of the relation

$$\left(\frac{\partial q}{\partial \psi}\right)_x = \left(\frac{\partial q}{\partial \psi}\right)_r + \left(\frac{\partial r}{\partial \psi}\right)_x \left(\frac{\partial q}{\partial r}\right)_\psi = \left(\frac{\partial q}{\partial \psi}\right)_r + \frac{\sqrt{q^2 + 1}}{2\pi\rho vr} \left(\frac{\partial q}{\partial r}\right)_\psi$$

can be written

$$\boxed{\pi\rho v \left(\frac{\partial q^2}{\partial \psi}\right)_r + \frac{1}{\sqrt{q^2 + 1}} \left(\frac{\partial q^2}{\partial r^2}\right)_\psi + \frac{q^2\sqrt{q^2 + 1}}{r^2} = 0} \tag{16}$$

To find the conditions for consistency of (14) and (16) we substitute the former into the latter and obtain

$$[4\pi\rho(\psi)v(\psi)\, dA(\psi)/d\psi + 3A^2(\psi)]r^4 + 4[\pi\rho(\psi)v(\psi)\, dB(\psi)/d\psi + A(\psi)B(\psi)]r^2 + [B^2(\psi) - 4] = 0$$

Two cases arise: (a) $B^2 = 4$ and $A = 0$. In this case (16) is satisfied identically. Equation (14) then requires that $q = 0$; i.e., that $r = r(\psi)$. (b) $B^2 \neq 4$ or $A \neq 0$. In this case (16) can be solved for r in terms of the coefficients. However all coefficients are functions of ψ alone, hence again $r = r(\psi)$. This establishes

THEOREM 2: *The only steady axially-symmetric flows of an inviscid fluid (in the absence of external force fields) in which the velocity magnitude, pressure, and density are constant along each individual streamline are purely axial flows.*

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(Received March 22, 1948)

COEFFICIENTS FOR REPEATED INTEGRATION WITH CENTRAL DIFFERENCES

By Herbert E. Salzer

The purpose of the present article is to facilitate the performance of a double or multiply repeated quadrature, with high accuracy, upon a function which is tabulated with its central differences of even order. A previous paper* furnished coefficients for the more universally applicable case of advancing and backward differences; but since central difference interpolation formulas converge more rapidly, integration formulas which are based upon them require no justification. The coefficients in the table below were obtained by repeated integration of the well-known Everett interpolation formula employing even central differences, namely,

$$f(x_0 + ph) \sim qf_0 + pf_1 + \sum_{s=1}^{m} \left[\frac{q(q^2 - 1^2)\cdots(q^2 - s^2)}{(2s+1)!}\, \delta_0^{2s} + \frac{p(p^2 - 1^2)\cdots(p^2 - s^2)}{(2s+1)!}\, \delta_1^{2s} \right],$$

where h is the tabular interval and $q = 1 - p$. The advantage in the present table is that it enables one to perform a k-fold repeated quadrature on $f(x)$ directly, whereas the alternative method is to obtain $1/(k-1)!$ times a single primitive of $(x-t)^{k-1}f(t)$, the latter process necessitating a series of multiplications and tabulation of a new set of differences for $(x-t)^{k-1}f(t)$. The advantage is increased if several different order primitives are required of the same function.

The coefficients given below are to be employed in stepwise k-fold quadrature of tabulated functions. They may also be used in the stepwise integration of higher order differential equations. A k-fold quadrature always introduces an arbitrary polynomial of the $(k-1)$th degree, which is determined by the value of the k-fold primitive at k nearby points, or the k-fold primitive and its first $(k-1)$ derivatives at a point. In stepwise integration, a particular k-fold primitive is obtained by making x_0 the lower limit of the repeated integral and x_1 the last upper limit, where $x_1 - x_0 = h$. Now suppose that in a k-fold quadrature problem one knows the desired k-fold primitive (denoted by ψ) and its derivatives at the point x_0. Then the k-fold particular integral up to x_1, obtained by formula (1) below plus a polynomial in $(x_1 - x_0)$ of the $(k-1)$th degree where the coefficient of $(x_1 - x_0)^s$ for $s = 0, 1, \cdots, k-1$ is $1/s!$ times the sth derivative at x_0, is equal to the value of ψ at x_1. Then one can find each derivative at x_1, the rth say, by using (1) with $(k-r)$ in place of k plus an obvious polynomial in $(x_1 - x_0)$, because the sth derivative of ψ at x_0 is the $(s-r)$th derivative of $\psi^{(r)}$. Thus, having found the values of the desired k-fold primitive ψ and its derivatives at x_1, we proceed to x_2 in exactly the same manner, with x_1 as the

* H. E. Salzer, "Table of Coefficients for Repeated Integration with Differences," Phil. Mag., Ser. 7, vol. xxxviii, pp. 331–338, 1947.

lower limit corresponding to the previous x_0, and x_2 corresponding to the previous x_1. Hence, a single-step formula is adequate for solving a variety of problems where a certain order derivative is tabulated and the function itself is required. Furthermore, the reason that only a single-step repeated integration formula is given here, is that the need for finding $\int_{x_0}^{x_m} \underset{k\text{-fold}}{\cdots} \int_{x_0}^{x} \int_{x_0}^{x} f(x)(dx)^k$ directly, is not comparable to that for the area under a curve in the case of a single quadrature.

The central-difference formula for repeated integration which is employed here is

$$(1) \qquad \int_{x_0}^{x_1} \underset{k\text{-fold}}{\cdots} \int_{x_0}^{x} \int_{x_0}^{x} f(x)(dx)^k = h^k \int_0^1 \underset{k\text{-fold}}{\cdots} \int_0^p \int_0^p f(x_0 + ph)(dp)^k$$

$$= h^k \left[A_0^{(k)} f_0 + B_0^{(k)} f_1 + \sum_{s=1}^{m} (A_{2s}^{(k)} \delta_0^{2s} + B_{2s}^{(k)} \delta_1^{2s}) \right] + R_{2m},$$

where the coefficients $A_{2s}^{(k)}$ and $B_{2s}^{(k)}$ are defined for $s \geq 1$ by

$$(2) \qquad A_{2s}^{(k)} = \frac{1}{(2s+1)!} \int_0^1 \underset{k\text{-fold}}{\cdots} \int_0^p \int_0^p q(q^2 - 1^2) \cdots (q^2 - s^2)(dp)^k,$$

where $q = 1 - p$, and

$$(3) \qquad B_{2s}^{(k)} = \frac{1}{(2s+1)!} \int_0^1 \underset{k\text{-fold}}{\cdots} \int_0^p \int_0^p p(p^2 - 1^2) \cdots (p^2 - s^2)(dp)^k,$$

with $A_0^{(k)} = 1/k! - 1/(k+1)!$ and $B_0^{(k)} = 1/(k+1)!$.

In the most important case, that of a single quadrature (and only then), $A_{2s}^{(1)} = B_{2s}^{(1)}$ and the quantities $M_{2s} \equiv 2A_{2s}^{(1)}$ (or $2B_{2s}^{(1)}$), occurring when (1) is written in the form of the Gauss-Encke integration formula, have been previously tabulated.[1] The tables in the present article give: (a) $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$, exact values for $2s = 0, 2, 4, \cdots, 20$ and sixteen decimal places for $2s = 22, 24, \cdots, 48$, the decimal values being guaranteed to well within a unit in the last place. (b) $A_{2s}^{(k)}$ and $B_{2s}^{(k)}$, for $k = 3, 4, 5$ and 6, and $2s = 0, 2, 4, \cdots, 20$, to eight significant figures, accurate to well within a unit in the last place. The greater extent of tabulation for $k = 2$ is due to the fact that problems involving double quadratures and second order differential equations are much more important by comparison with higher order problems.

The quantities $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$ are related to

$$M_{2s} \equiv \frac{2}{(2s+1)!} \int_0^1 t(t^2 - 1^2) \cdots (t^2 - s^2)dt \text{ as follows:}$$

[1] H. E. Salzer, "Coefficients for Numerical Integration with Central Differences," Phil. Mag., Ser. 7, vol. xxxv, pp. 262–264, 1944. This paper contains exact values through M_{28} and eighteen decimals through M_{16}.

$$A_{2s}^{(2)} = [(s+1)/2]M_{2s} + (2s+2)\,M_{2s+2}\,, \tag{4}$$

$$B_{2s}^{(2)} = -\,(s/2)M_{2s} - (2s+2)\,M_{2s+2}\,. \tag{5}$$

Coefficients of δ_0^{2s} and δ_1^{2s} in the double quadrature formula

$$\int_{x_0}^{x_1}\int_{x_0}^{x} f(x)(dx)^2 = h^2\left[A_0^{(2)}f_0 + B_0^{(2)}f_1 + \sum_{s=1}^{m}\left(A_{2s}^{(2)}\,\delta_0^{2s} + B_{2s}^{(2)}\,\delta_1^{2s}\right)\right] + R_{2m}$$

$2s$	$A_{2s}^{(2)}$	$B_{2s}^{(2)}$
0	$\frac{1}{3}$	$\frac{1}{6}$
2	$-\frac{1}{45}$	$-\frac{7}{360}$
4	$\frac{1}{252}$	$\frac{37}{10080}$
6	$-\frac{23}{28350}$	$-\frac{199}{2\ 59200}$
8	$\frac{263}{14\ 96880}$	$\frac{40321}{2395\ 00800}$
10	$-\frac{1\ 33787}{34054\ 02000}$	$-\frac{55\ 12813}{14\ 52971\ 52000}$
12	$\frac{1\ 57009}{1\ 75134\ 96000}$	$\frac{1366\ 01407}{1569\ 20924\ 16000}$
14	$-\frac{162\ 15071}{781\ 53975\ 90000}$	$-\frac{3\ 23735\ 35937}{16\ 00593\ 42643\ 20000}$
16	$\frac{26894\ 53969}{5\ 54372\ 20238\ 40000}$	$\frac{403\ 93141\ 45093}{8515\ 15702\ 86182\ 40000}$
18	$-\frac{2\ 68931\ 18531}{235\ 21220\ 58686\ 40000}$	$-\frac{37788\ 04671\ 85583}{33\ 72002\ 18333\ 28230\ 40000}$
20	$\frac{560\ 07519\ 28169}{2\ 06559\ 08260\ 82784\ 00000}$	$\frac{123\ 90511\ 32655\ 94071}{46533\ 63012\ 99929\ 57952\ 00000}$
22	$-.00000\ 00064\ 62633\ 7$	$-.00000\ 00063\ 56509\ 8$
24	$.00000\ 00015\ 46760\ 4$	$.00000\ 00015\ 23387\ 3$
26	$-.00000\ 00003\ 71495\ 2$	$-.00000\ 00003\ 66296\ 3$
28	$.00000\ 00000\ 89490\ 1$	$.00000\ 00000\ 88323\ 9$
30	$-.00000\ 00000\ 21612\ 8$	$-.00000\ 00000\ 21349\ 2$
32	$.00000\ 00000\ 05231\ 4$	$.00000\ 00000\ 05171\ 4$
34	$-.00000\ 00000\ 01268\ 7$	$-.00000\ 00000\ 01255\ 0$
36	$.00000\ 00000\ 00308\ 2$	$.00000\ 00000\ 00305\ 1$
38	$-.00000\ 00000\ 00075\ 0$	$-.00000\ 00000\ 00074\ 3$
40	$.00000\ 00000\ 00018\ 3$	$.00000\ 00000\ 00018\ 1$
42	$-.00000\ 00000\ 00004\ 5$	$-.00000\ 00000\ 00004\ 4$
44	$.00000\ 00000\ 00001\ 1$	$.00000\ 00000\ 00001\ 1$
46	$-.00000\ 00000\ 00000\ 3$	$-.00000\ 00000\ 00000\ 3$
48	$.00000\ 00000\ 00000\ 1$	$.00000\ 00000\ 00000\ 1$

To establish (4), consider $A_{2s}^{(2)} = \dfrac{1}{(2s+1)!}\displaystyle\int_0^1\int_0^p q(q^2-1^2)\cdots(q^2-s^2)(dp)^2$

Coefficients of δ_0^{2s} and δ_1^{2s} in the k-fold quadrature formula

$$\int_{x_0}^{x_1} \cdots \int_{x_0}^{x} \int_{x_0}^{x} f(x)(dx)^k = h^k \left[A_0^{(k)} f_0 + B_0^{(k)} f_1 + \sum_{s=1}^{m} (A_{2s}^{(k)} \delta_0^{2s} + B_{2s}^{(k)} \delta_1^{2s}) \right] + R_{2m}$$

(k-fold)

$2s$	$k = 3$		$k = 4$	
	$A_{2s}^{(3)}$	$B_{2s}^{(3)}$	$A_{2s}^{(4)}$	$B_{2s}^{(4)}$
0	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{30}$	$\frac{1}{120}$
2	−.00694 44444	−.00555 55556	−.00158 73016	−.00119 04762
4	.00121 52778	.00106 64683	.00027 33686 1	.00023 09303 4
6	−.00024 63624 3	−.00022 45921 5	−.00005 50478 33	−.00004 88432 61
8	.00005 31052 51	.00004 94330 29	.00001 18221 48	.00001 07758 98
10	−.00001 18392 54	−.00001 11667 27	−.00000 26294 334	−.00000 24378 618
12	.00000 26961 137	.00000 25661 547	.00000 05978 1851	.00000 05608 0404
14	−.00000 06230 1738	−.00000 05969 3551	−.00000 01379 7901	−.00000 01305 5115
16	.00000 01455 1069	.00000 01401 2693	.00000 00321 96698	.00000 00306 63564
18	−.00000 00342 62474	−.00000 00331 26757	−.00000 00075 75688 0	−.00000 00072 52287 5
20	.00000 00081 19322 9	.00000 00078 75560 9	.00000 00017 94195 1	.00000 00017 24785 8

Coefficients of δ_0^{2s} and δ_1^{2s} in the k-fold quadrature formula

$$\int_{x_0}^{x_1} \underbrace{\cdots}_{k\text{-fold}} \int_{x_0}^{x} \int_{x_0}^{x} f(x)(dx)^k = h^k \left[A_0^{(k)} f_0 + B_0^{(k)} f_1 + \sum_{i=1}^{m} (A_{2s}^{(k)} \delta_0^{2s} + B_{2s}^{(k)} \delta_1^{2s}) \right] + R_{2m}$$

	$k = 5$		$k = 6$	
$2s$	$A_{2s}^{(5)}$	$B_{2s}^{(5)}$	$A_{2s}^{(6)}$	$B_{2s}^{(6)}$
0	$\frac{1}{144}$	$\frac{1}{720}$	$\frac{1}{840}$	$\frac{1}{5040}$
2	−.00028 93518 5	−.00020 66798 9	−.00004 40917 11	−.00003 03130 51
4	.00004 91898 15	.00004 03714 73	.00000 74154 241	.00000 59498 757
6	−.00000 98517 416	−.00000 85636 457	−.00000 14785 332	−.00000 12646 726
8	.00000 21094 472	.00000 18923 404	.00000 03158 0135	.00000 02797 7352
10	−.00000 04682 8244	−.00000 04285 4048	−.00000 00699 95719	−.00000 00634 02727
12	.00000 01063 2683	.00000 00986 49508	.00000 00158 75760	.00000 00146 02389
14	−.00000 00245 16969	−.00000 00229 76530	−.00000 00036 57734 9	−.00000 00034 02272 5
16	.00000 00057 16681 2	.00000 00053 98761 1	.00000 00008 52362 88	.00000 00007 99645 54
18	−.00000 00013 44315 1	−.00000 00012 77258 0	−.00000 00002 00342 19	−.00000 00001 89223 74
20	.00000 00003 18231 59	.00000 00003 03840 44	.00000 00000 47407 291	.00000 00000 45021 322

$= \frac{1}{(2s+1)!}\int_0^1 (1-p)q(q^2-1^2)\cdots(q^2-s^2)dp$. Since $1-p=q$,

$A_{2s}^{(2)} = \frac{1}{(2s+1)!}\int_0^1 q^2(q^2-1^2)\cdots(q^2-s^2)dq$, which is rewritten as

$\frac{1}{(2s+1)!}\int_0^1 (q-s-1+s+1)\,q\,(q^2-1^2)\cdots(q^2-s^2)dq$, or

$$A_{2s}^{(2)} = [(s+1)/2]M_{2s} + \frac{2s+2}{(2s+2)!}\int_0^1 q(q^2-1^2)\cdots(q^2-s^2)(q-[s+1])dq.$$

But it is also true that

(A) $M_{2s} = \frac{1}{(2s)!}\int_0^1 (q+s-1)(q+s-2)\cdots(q-s)\,dq$ (M.[2] p. 184–185),

so that the second term of the last expression for $A_{2s}^{(2)}$ is seen to be $(2s+2)M_{2s+2}$, which proves (4). In similar fashion, (5) follows from $B_{2s}^{(2)} = \frac{1}{(2s+1)!}$ $\int_0^1\int_0^p p(p^2-1^2)\cdots(p^2-s^2)(dp)^2 = \frac{1}{(2s+1)!}\int_0^1 (1-p)p(p^2-1^2)\cdots$ $(p^2-s^2)dp$, which is rewritten $B_{2s}^{(2)} = \frac{-1}{(2s+1)!}\int_0^1 (p-s-1+s)p(p^2-1^2)\cdots$ $(p^2-s^2)dp$, and using (A) again, this last expression is seen to be $-(s/2)M_{2s} - (2s+2)M_{2s+2}$. From (4) and (5) one has the convenient checking formula

$$A_{2s}^{(2)} + B_{2s}^{(2)} = \tfrac{1}{2}M_{2s} = A_{2s}^{(1)}, \tag{6}$$

which is a special case of a more general recursion formula connecting $B_{2s}^{(k)}$ and $A_{2s}^{(i)}$ for $i = 1, \cdots, k$. (See (10) below.)

Using (6), one may express the general term in (1) for the case $k = 2$, in an equivalent alternative form, namely,

$$A_{2s}^{(2)}\delta_0^{2s} + B_{2s}^{(2)}\delta_1^{2s} = A_{2s}^{(1)}\delta_0^{2s} + B_{2s}^{(2)}\delta_1^{2s+1}, \tag{B}$$

which is analogous to the forward version of the Newton-Gauss formula for interpolation.

The quantities $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$ have rather simple expressions in terms of $B_\nu^{(n)}(x)$, Bernoulli polynomials of degree ν and order n, defined by the relation

$$t^n e^{xt}/(e^t-1)^n = \textstyle\sum_{\nu=0}^{\infty} t^\nu B_\nu^{(n)}(x)/\nu! \qquad \text{(M. p. 127(2) upper).}$$

($B_{2s}^{(2)}$ in this notation is not to be confused with Milne-Thomson's $B_\nu^{(n)}$ for $B_\nu^{(n)}(0)$.) Thus for

$$A_{2s}^{(2)} = \frac{1}{(2s+1)!}\int_0^1\int_0^p q(q^2-1^2)\cdots(q^2-s^2)(dp)^2, \quad \text{where } q = 1-p,$$

[2] M. here denotes L. M. Milne-Thomson, "Calculus of Finite Differences," Macmillan 1933.

in the inner quadrature change the variable to q to get

$$\int_{1-p}^{1} q(q^2 - 1^2) \cdots (q^2 - s^2)dq.$$

But since

$$q(q^2 - 1^2) \cdots (q^2 - s^2) = B_{2s+1}^{2s+2}(q + s + 1) \qquad \text{(M.p. 130(1))},$$

by letting $q + s + 1 = r$, the inner integral is now $\int_{s+2-p}^{s+2} B_{2s+1}^{2s+2}(r)dr$, which is also equal to $\frac{1}{2s+2}[B_{2s+2}^{(2s+2)}(s + 2) - B_{2s+2}^{(2s+2)}(s + 2 - p)]$ (M. p. 127(3) lower). Now the integration between 0 and 1, after changing the variable from p to $s + 2 - p$ and making use of

(C) $$\int_{s}^{s+1} B_{s}^{(n)}(t)\, dt = B_{s}^{(n-1)}(x) \qquad \text{(M. p. 128(9))}, \text{ yields}$$

(7) $$A_{2s}^{(2)} = \frac{1}{(2s+2)!}[B_{2s+2}^{(2s+2)}(s + 2) - B_{2s+2}^{(2s+1)}(s + 1)].$$

In similar fashion, from $B_{2s}^{(2)} = \frac{1}{(2s+1)!}\int_0^1 \int_0^p p(p^2 - 1^2) \cdots (p^2 - s^2)(dp)^2$, one obtains $\frac{1}{(2s+1)!}\int_0^1 \int_0^p B_{2s+1}^{2s+2}(p + s + 1)(dp)^2$ (M. p. 130(1)). Thus $B_{2s}^{(2)} = \frac{1}{(2s+1)!}\int_0^1 \int_{s+1}^{p+s+1} B_{2s+1}^{(2s+2)}(r)\, dr\, dp = \frac{1}{(2s+1)!}\int_0^1 \frac{1}{(2s+2)}[B_{2s+2}^{(2s+2)}(p + s + 1) - B_{2s+2}^{(2s+2)}(s + 1)]dp$ (M. p. 127(3) lower), or, by letting $p + s + 1$ be the new variable of integration and using (C),

(8) $$B_{2s}^{(2)} = \frac{1}{(2s+2)!}[B_{2s+2}^{(2s+1)}(s + 1) - B_{2s+2}^{(2s+2)}(s + 1)].$$

For any value of $k \geq 1$, the following cumulative recursion formulas hold:

(9) $$A_{2s}^{(k)} = \frac{1}{(k-1)!}B_{2s}^{(1)} - \frac{1}{(k-2)!}B_{2s}^{(2)} + \cdots + \frac{(-1)^{k-2}}{1!}B_{2s}^{(k-1)} + \frac{(-1)^{k-1}}{0!}B_{2s}^{(k)},$$

(10) $$B_{2s}^{(k)} = \frac{1}{(k-1)!}A_{2s}^{(1)} - \frac{1}{(k-2)!}A_{2s}^{(2)} + \cdots + \frac{(-1)^{k-2}}{1!}A_{2s}^{(k-1)} + \frac{(-1)^{k-1}}{0!}A_{2s}^{(k)}.$$

To prove (9), consider

$$\int_0^1 \cdots \int_0^p \int_0^p \phi(q)(dp)^k = \frac{1}{(k-1)!}\int_0^1 (1 - p)^{k-1}\phi(q)\, dp$$

(k-fold)

$$= \frac{1}{(k-1)!}\int_0^1 \sum_{i=0}^{k-1}\binom{k-1}{i}(-p)^i\phi(q)\,dp$$

$$= \sum_{i=0}^{k-1}\frac{1}{(k-1)!}\binom{k-1}{i}\int_0^1 (q-1)^i\phi(q)\,dq$$

$$= \sum_{i=0}^{k-1}\frac{i!}{i!}\binom{k-1}{i}(-1)^i\frac{1}{(k-1)!}\int_0^1 (1-q)^i\phi(q)\,dq$$

$$= \sum_{i=0}^{k-1}\binom{k-1}{i}\frac{i!}{(k-1)!}(-1)^i\underbrace{\int_0^1\cdots\int_0^q}_{(i+1)\text{-fold}}\int_0^q \phi(q)(dq)^{i+1}.$$

But $\binom{k-1}{i}\frac{i!}{(k-1)!} = \frac{1}{(k-i-1)!}$, and by setting $\phi(q) = q(q^2-1^2)\cdots(q^2-s^2)$, multiplying through by $1/(2s+1)!$, and from the definitions of $A_{2s}^{(k)}$ and $B_{2s}^{(k)}$ in (2) and (3), equation (9) is obtained. Equation (10) can be established in almost exactly the same manner.

The coefficients $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$ were calculated by the recursion formulas (4) and (5), and checked by (6). The coefficients $A_{2s}^{(k)}$ and $B_{2s}^{(k)}$, $k = 3(1)6$, were calculated directly by repeated integration of the central factorial polynomials $q(q^2-1^2)\cdots(q^2-s^2)$ expressed as polynomials in p, and $p(p^2-1^2)\cdots(p^2-s^2)$, according to (2) and (3), and they were checked by both (9) and (10), with an agreement to within a half-unit in the eighth significant figure.

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(Received March 12, 1948)

THE CONSISTENCY RELATIONS FOR SHOCK WAVES*

By T. Y. Thomas

1. Introduction. We deal with the plane steady flow of a gas in which a shock wave or *shock line* exists, e.g. a shock line originating at the vertex V of a pointed obstacle in the field of the flow. The usual assumptions that the flow is irrotational and adiabatic behind the shock line will not be made, but viscosity and thermal conductivity will be neglected. Continuity and differentiability requirements on the functions involved in the discussion will be obvious from the demands of the analysis and will not be stated explicitly.

This paper is a continuation of two previous ones: the article *On Curved Shock Waves* [1] and the article entitled *Calculation of the Curvatures of Attached Shock Waves* [2]. The notation will be consistent with that previously employed. In the first of these papers we showed that the conditions of shock which can be written in the form

$$[u_\alpha] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_\alpha}{-(\gamma+1)\rho_1 u_{1n}}, \tag{1}$$

$$[p] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{\gamma+1}, \tag{2}$$

$$[\rho] = \frac{2\rho_1(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma-1)\rho_1 u_{1n}^2}, \tag{3}$$

and the differential equations governing the flow on both sides of the shock line, namely

$$p_{,\alpha} + \rho u_\sigma u_{\alpha,\sigma} = 0, \tag{4}$$

$$\rho_{,\alpha} u_\alpha + \rho u_{\sigma,\sigma} = 0, \tag{5}$$

$$\rho u_{\alpha,\beta} u_\alpha u_\beta - \gamma p u_{\sigma,\sigma} = 0, \tag{6}$$

lead to the determination of the derivatives $u_{\alpha,\beta}$, $p_{,\alpha}$ and $\rho_{,\alpha}$ at any point immediately behind the shock line when the values of the quantities u_α, p and ρ and their first derivatives are known at the corresponding point in front of the shock line together with the direction λ and the curvature κ of the shock line at this point. Quantities with subscript 1 in the above relations (and following relations) refer to values before the shock line and quantities without this subscript to quantities behind the shock line. The determination of the derivatives was given in the form

$$u_{\alpha,\beta} = A_{ij}\zeta_\alpha^i \zeta_\beta^j; \qquad p_{,\alpha} = -\rho A_{ij}\zeta_\alpha^i; \qquad \rho_{,\alpha} = a_i \zeta_\alpha^i \tag{7}$$

where the A_{ij} and a_i are invariants of the flow with values along the shock line. If we put $\xi_1^\alpha = \lambda^\alpha$ and $\xi_2^\alpha = u^\alpha$ where λ is the unit tangent vector to the shock line

* Prepared under Navy Contract N6onr-180, Task Order No. V with Indiana University.

and the u^a are the velocity components immediately behind the shock, the quantities ζ^i_a are defined by the equations $\zeta^i_a \xi^a_j = \delta^i_j$. The explicit evaluation of the ζ^i_a in matrix notation is given by

$$\| \zeta^i_a \| = \begin{Vmatrix} \zeta^1_1 & \zeta^2_1 \\ \zeta^1_2 & \zeta^2_2 \end{Vmatrix} = \begin{Vmatrix} u^2/u_n & -\lambda^2/u_n \\ -u^1/u_n & \lambda^1/u_n \end{Vmatrix}$$

where u_n is the normal component of velocity immediately behind the shock line.

For uniform incident flow the A_{ij} and a_i were calculated in explicit form and found to be given by the equations

$$A_{11} = u_n[\rho]\kappa/\rho_1 , \tag{8}$$

$$A_{12} = 4u_n u_t \kappa/(\gamma + 1), \tag{9}$$

$$A_{21} = \left(\frac{[\rho]^2}{\rho_1 \rho} + \frac{4}{\gamma + 1}\right) u_n u_t \kappa, \tag{10}$$

$$A_{22} = \frac{\gamma p u_n \kappa}{\rho u_n^2 - \gamma p}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1 \rho} + \frac{8}{\gamma + 1}\right) u_t^2\right\}, \tag{11}$$

where v is the magnitude of the velocity and u_t its tangential component immediately behind the shock line; also

$$a_1 = \frac{2u_t}{(\gamma + 1)u_n}\{(\gamma - 1)\rho - (\gamma + 1)\rho_1\}\kappa, \tag{12}$$

$$a_2 = \frac{-\rho^2 u_n \kappa}{\rho u_n^2 - \gamma p}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1 \rho} + \frac{8}{\gamma + 1}\right) u_t^2\right\}. \tag{13}$$

Denoting the arc length along the shock line by s it was also found, for uniform incident flow, that

$$du_a/ds = A_{ai}\zeta^i_a , \tag{14}$$

$$d\rho/ds = a_1 , \tag{15}$$

$$dp/ds = -4\rho_1 u_{1n} u_t \kappa/(\gamma + 1), \tag{16}$$

$$v dv/ds = A_{21} . \tag{17}$$

On the basis of these results it was shown that when the flow before the shock line was uniform the relation between the curvature κ at any point of the shock line and the curvature K of the stream line at this point behind the shock line is given by

$$K = -\left\{\frac{\left(\frac{\rho}{\rho_1} + \frac{\rho_1}{\rho} + \frac{6 - 2\gamma}{\gamma + 1}\right)\left(\frac{u_t}{v}\right)^3 - \left(\frac{\rho}{\rho_1} - 1\right)\left(\frac{u_t}{v}\right)}{\frac{u_n^2}{c^2} - 1} + \frac{4}{\gamma + 1}\left(\frac{u_t}{v}\right)\right\}\kappa, \tag{18}$$

where c, defined by $c^2 = \gamma p/\rho$, is the velocity of sound in the gas.

2. Consistency relations. Denote by η the arc length measured in the direction of flow along any stream line behind the shock line. Put

$$\frac{d^n K}{d\eta^n} = K_{(n)}, \qquad \frac{d^n \kappa}{ds^n} = \kappa_{(n)}, \qquad (n = 0, 1, 2, \cdots),$$

so that $K_{(0)} = K$ and $\kappa_{(0)} = \kappa$. It is then clear from equation (18) and the discussion of §7 of the present paper that for uniform incident flow, and for the above values of n, there exist relations of the form

$$K_{(n)} = G_n(M, \alpha)\kappa_{(n)} + H_n, \tag{19}$$

where G_n depends on the mach number M of the incident flow and the inclination α of the shock line; the quantity H_n is a polynomial in κ and its derivatives with respect to s of order less than n and the coefficients of this polynomial are functions of α and M. For $n = 0$ the relation (19) is identical with (18) so that $H_0 = 0$ and G_0 is the negative of the bracket expression in (18). In §6 we have calculated the expression for G_1; from the discussion of this paragraph and of §7, it is seen that H_1 is of the form $h(M, \alpha)\kappa^2$.

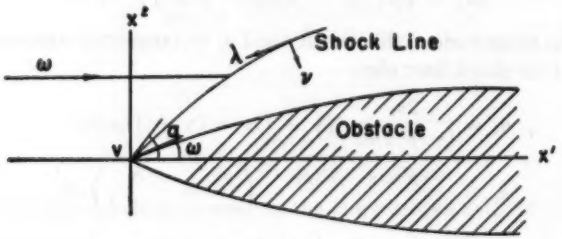

FIG. 1

The relations (19) for $n = 0, 1, 2, \cdots$ will be called the *consistency relations for plane shock waves*. These relations are of particular interest when considered at the vertex V of a pointed obstacle in the field of flow; we have illustrated this situation in the following figure for an incident flow of magnitude w having a direction parallel to the positive direction of the x^1 axis. At V the values of the quantities $K_{(n)}$ are imposed by the boundary conditions so that there is the possibility that any one of the relations (19) cannot be satisfied when the quantities $K_{(n)}$ and $\kappa_{(n)}$ are continuous functions of the arc lengths η and s respectively. In such a case the shock line solution of the problem will fail to exist under the conditions of differentiability involved in the derivation of the inconsistent relation (19).

3. Singular shock directions. The values of α at V which are initially allowable satisfy a condition of the form $\alpha_0(M) < \alpha \leqq 90°$ [2]. For a given value of $M > 1$, a value of α in the above range for which any one of the $G_n(M, \alpha) = 0$ will be called *singular relative to* M. It is evident that these singular values of α have an exceptional role in the general shock line solution of the hydrodynamical problem.

4. Calculation of second derivatives behind the shock line. The discussion in this section is general and independent of the condition of uniform flow before the shock line. The procedure is analogous to that used in the calculation of first derivatives behind the shock lines [1]. We seek expressions for the second derivatives of the form

$$u_{\alpha,\beta\gamma} = I_{ijk}\zeta^i_\alpha \zeta^j_\beta \zeta^k_\gamma , \tag{20}$$

$$p_{,\beta\gamma} = J_{jk}\zeta^j_\beta \zeta^k_\gamma , \tag{21}$$

$$\rho_{,\beta\gamma} = K_{jk}\zeta^j_\beta \zeta^k_\gamma , \tag{22}$$

where the I, J and K are (scalar) invariants of the flow with values along the shock line. These invariants can be thought of as defined by the following equations, equivalent to (20), (21) and (22), namely

$$I_{ijk} = u_{\alpha,\beta\gamma}\xi^\alpha_i \xi^\beta_j \xi^\gamma_k , \qquad \text{(symmetric in } j, k), \tag{23}$$

$$J_{ij} = p_{,\alpha\beta}\xi^\alpha_i \xi^\beta_j , \qquad \text{(symmetric in } i, j), \tag{24}$$

$$K_{ij} = \rho_{,\alpha\beta}\xi^\alpha_i \xi^\beta_j , \qquad \text{(symmetric in } i, j). \tag{25}$$

Differentiation of $u_{\alpha,\beta}$, $p_{,\beta}$ and $\rho_{,\beta}$ with respect to s, gives

$$u_{\alpha,\beta\gamma}\xi^\gamma_1 = P_{\alpha\beta} , \tag{26}$$

$$p_{,\beta\gamma}\xi^\gamma_1 = Q_\beta , \tag{27}$$

$$\rho_{,\beta\gamma}\xi^\gamma_1 = R_\beta . \tag{28}$$

Here P, Q and R are tensors whose components are found along the shock line by differentiating the right members of equations (7) with respect to s. The determination of these tensors is *allowable* in the sense that their components can be calculated at any point of the shock line in terms of the quantities u_α, p and ρ and their first and second derivatives immediately before the shock line, its curvature κ and the derivative $d\kappa/ds$ at the point. From equations (23), (24), (25) and (26), (27), (28) we can write

$$I_{ij1} = u_{\alpha,\beta\gamma}\xi^\alpha_i \xi^\beta_j \xi^\gamma_1 = P_{\alpha\beta}\xi^\alpha_i \xi^\beta_j , \tag{29}$$

$$J_{i1} = p_{,\beta\gamma}\xi^\beta_i \xi^\gamma_1 = Q_\beta \xi^\beta_i , \tag{30}$$

$$K_{i1} = \rho_{,\beta\gamma}\xi^\beta_i \xi^\gamma_1 = R_\beta \xi^\beta_i . \tag{31}$$

Hence the invariants I_{ij1}, J_{i1} and K_{i1} have an allowable determination. It remains to find allowable determinations for the other invariants I, J and K; explicitly these are I_{i22}, J_{22} and K_{22} in view of the symmetry conditions satisfied by these invariants.

To obtain the allowable determinations of I_{i22}, J_{22} and K_{22} we must have recourse to the system of equations which results by differentiation of (4), (5) and (6). These equations are

$$p_{,\alpha\beta} + \rho u_\sigma u_{\alpha,\sigma\beta} + (\rho u_\sigma)_{,\beta} u_{\alpha,\sigma} = 0, \tag{32}$$

$$\rho_{,\alpha\beta}u_\alpha + \rho_{,\alpha}u_{\alpha,\beta} + \rho_{,\beta}u_{\sigma,\sigma} + \rho u_{\sigma,\sigma\beta} = 0, \tag{33}$$

$$\rho u_{\alpha,\beta\gamma}u_\alpha u_\beta + \rho_{,\gamma}u_{\alpha,\beta}u_\alpha u_\beta + \rho u_{\alpha,\beta}u_{\alpha,\gamma}u_\beta + \rho u_{\alpha,\beta}u_\alpha u_{\beta,\gamma} - \gamma p_{,\gamma}u_{\sigma,\sigma} - \gamma p u_{\sigma,\sigma\gamma} = 0. \tag{34}$$

Now multiply (32) by $\xi_i^\alpha \xi_j^\beta$ and make use of (23) and (24) to obtain

$$\rho I_{ij2} + J_{ij} + (\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}\xi_i^\alpha \xi_j^\beta = 0,$$

or, when written in full, we have

$$\rho I_{112} + J_{11} + (\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}\lambda^\alpha\lambda^\beta = 0, \tag{35}$$

$$\rho I_{122} + J_{12} + (\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}\lambda^\alpha u^\beta = 0, \tag{36}$$

$$\rho I_{212} + J_{21} + (\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}u^\alpha\lambda^\beta = 0, \tag{37}$$

$$\rho I_{222} + J_{22} + (\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}u^\alpha u^\beta = 0. \tag{38}$$

The first and third of these equations give relations between invariants I and J for which allowable determinations are known while (36) gives the allowable determination of I_{122} ; equation (38) relates the invariants I_{222} and J_{22} for which allowable determinations must be found. Again multiplying (34) by u_γ we have

$$\rho I_{222} + \rho_{,\gamma}u_{\alpha,\beta}u_\alpha u_\beta u_\gamma + \rho u_{\alpha,\beta}u_{\alpha,\gamma}u_\beta u_\gamma + \rho u_{\alpha,\beta}u_{\beta,\gamma}u_\alpha u_\gamma - \gamma p_{,\gamma}u_{\sigma,\sigma}u_\gamma - \gamma p u_{\sigma,\sigma\gamma}u_\gamma = 0. \tag{39}$$

Subtracting corresponding members of (38) and (39) the resulting equation can be written in the form

$$J_{22} = \rho_{,\gamma}u_{\alpha,\beta}u_\alpha u_\beta u_\gamma + \rho u_{\alpha,\beta}u_{\alpha,\gamma}u_\beta u_\gamma + \frac{(\gamma+1)}{\rho}p_{,\alpha}\rho_{,\beta}u_\alpha u_\beta - \gamma p u_{\sigma,\sigma\gamma}u_\gamma. \tag{40}$$

But from (20) we have

$$u_{\sigma,\sigma\gamma}u_\gamma = I_{ijk}\xi_\sigma^i\xi_\sigma^j\xi_\gamma^k\xi_\gamma^7 = I_{ij2}\xi_\sigma^i\xi_\sigma^j.$$

When we expand the right member of this equation, eliminate the invariants I_{112} , I_{122} and I_{212} by means of (35), (36) and (37) and then substitute for J_{11} and $J_{12} = J_{21}$ from (30) and for J_{22} from (40) we can show that the resulting equation can be written

$$(c^2 - u_n^2)u_{\sigma,\sigma\gamma}u_\gamma = \frac{v^2}{\rho}Q_\beta\lambda^\beta - \frac{2u_l}{\rho}Q_\beta u^\beta + \frac{1}{\rho}\rho_{,\gamma}u_{\alpha,\beta}u^\alpha u^\beta u^\gamma + u_{\alpha,\beta}u_{\alpha,\gamma}u^\beta u^\gamma + \frac{(\gamma+1)}{\rho^2}\rho_{,\alpha}\rho_{,\beta}u^\alpha u^\beta + \frac{1}{\rho}[v^2\lambda^\alpha\lambda^\beta - u_l(\lambda^\alpha u^\beta + u^\alpha\lambda^\beta) + u^\alpha u^\beta](\rho u_\sigma)_{,\beta}u_{\alpha,\sigma}. \tag{41}$$

Finally substituting the expression for $u_{\sigma,\sigma\gamma}u_\gamma$ given by (41) into equation (40) we find after some reduction that

$$J_{22} = \frac{c^2}{u_n^2 - c^2}\,[v^2 Q_\beta \lambda^\beta - 2u_i Q_\beta u^\beta + u_n^2(\rho u_\sigma)_{,\alpha} u_{\alpha,\sigma}]$$

(42)

$$+ \frac{u_n^2}{u_n^2 - c^2}\left[\frac{\gamma}{\rho}\, p_{,\alpha}\rho_{,\beta}u^\alpha u^\beta + \frac{p_{,\sigma}p_{,\sigma}}{\rho}\right].$$

This furnishes the allowable determination for J_{22}. Using this determination of J_{22} the allowable determination of I_{222} is given by (38).

To find the allowable determination of the remaining invariant K_{22} we multiply (33) by u_β and then apply (20) and (25) to obtain

$$K_{22} = -\rho_{,\alpha}u_{\alpha,\beta}u_\beta - u_{\sigma,\sigma}\rho_{,\beta}u_\beta - \rho I_{jk2}\zeta_\sigma^j\zeta_\sigma^k. \tag{43}$$

When we substitute into (43) the allowable determinations of the I_{jk2} this equation gives us the allowable determination of K_{22}. This completes the allowable determinations of all invariants I, J and K. When these determinations are used in the equations (20), (21) and (22) we obtain the allowable determinations of the second derivatives of the quantities u_α, p and ρ behind the shock line.

5. Uniform incident flow. In §1 we have listed the formulas (8), $\cdots$, (13) for the invariants A_{ij} and a_i when the velocity, pressure and density are constant before the shock (uniform incident flow). When taken in conjunction with the shock conditions (1), (2) and (3) these formulas give the explicit expressions for the allowable determinations of the A_{ij} and a_i. We now seek corresponding expressions for the invariants I, J and K under the assumption of uniform incident flow.

The following list of formulas will be found to be useful in our calculations; they can easily be derived on the basis of the considerations in §1. These formulas are

$$\lambda_\alpha = e_{\alpha\beta}\nu^\beta, \qquad \frac{d\lambda_\alpha}{ds} = \kappa\omega_\alpha, \qquad \frac{d\nu_\alpha}{ds} = -\kappa\lambda_\alpha$$

$$\zeta_\beta^i = \frac{e_{\beta\gamma}e_{ij}\xi_j^\gamma}{u_n}, \quad \text{or} \quad \zeta_\beta^1 = \frac{-e_{\alpha\beta}u^\alpha}{u_n}, \quad \zeta_\beta^2 = \frac{e_{\alpha\beta}\lambda^\alpha}{u_n}$$

$$\zeta_\alpha^1\nu^\alpha = -\frac{e_{\alpha\beta}u^\alpha u^\beta}{b_n} = -\frac{(u_1\nu_2 - u_2\nu_1)}{u_n} = -\frac{u_t}{u_n}$$

$$\zeta_\alpha^2\nu^\alpha = \frac{e_{\alpha\beta}\lambda^\alpha\nu^\beta}{u_n} = \frac{\lambda_1\nu_2 - \lambda_2\nu_1}{u_n} = \frac{1}{u_n}$$

$$\frac{d\xi_1^\gamma}{ds} = \frac{d\lambda^\gamma}{ds} = \kappa\nu^\gamma, \qquad \frac{d\xi_2^\gamma}{ds} = \frac{du^\gamma}{ds} = A_{\alpha\beta}\zeta_\gamma^i.$$

Now differentiate the relation $u_n = u_\alpha \nu^\alpha$ with respect to s, obtaining

$$\begin{aligned}\frac{du_n}{ds} &= \frac{du_\alpha}{ds}\nu^\alpha + u_\alpha \frac{d\nu^\alpha}{ds} = u_{\alpha,\beta}\nu^\alpha\lambda^\beta - \kappa u_\alpha\lambda^\alpha \\ &= A_{ij}\zeta^i_\alpha\zeta^j_\beta\xi^\beta_1\nu^\alpha - \kappa u_t = A_{\alpha 1}\zeta^i_\alpha\nu^\alpha - \kappa u_t \\ &= \frac{A_{21} - A_{11}u_t}{u_n} - \kappa u_t .\end{aligned} \tag{44}$$

In a similar manner by differentiation of the relation $u_{1n} = u_{1\alpha}\nu^\alpha$, where u_{1n} and $u_{1\alpha}$ are the normal velocity and the velocity components respectively before the shock line, we find

$$\frac{du_{1n}}{ds} = u_{1\alpha}\frac{d\nu^\alpha}{ds} = -u_{1\alpha}\kappa\lambda^\alpha = -\kappa u_t . \tag{45}$$

Also from $u_t = u_{1\alpha}\lambda^\alpha$, we have

$$\frac{du_t}{ds} = u_{1\alpha}\kappa\nu^\alpha = \kappa u_{1n} . \tag{46}$$

Again,

$$\frac{d\zeta^1_\beta}{ds} = -\frac{d}{ds}\left(\frac{e_{\alpha\beta}u^\alpha}{u_n}\right) = -\frac{e_{\alpha\beta}A_{\alpha 1}\zeta^i_\alpha}{u_n} + \frac{e_{\alpha\beta}u^\alpha}{u_n^2}\frac{du_n}{ds}, \tag{47}$$

$$\frac{d\zeta^2_\beta}{ds} = \frac{d}{ds}\left(\frac{e_{\alpha\beta}\lambda^\alpha}{u_n}\right) = -\frac{\kappa\lambda_\beta}{u_n} - \frac{\nu_\beta}{u_n^2}\frac{du_n}{ds}, \tag{48}$$

where the expression for du_n/ds in (47) and (48) is given by (44).

Equations (15), (16), (17) together with the above equations (44) and (46) provide us with the formulas needed in connection with the differentiation, with respect to s, of the right members of (8), $\cdots$, (13). However, it will be useful to have the derivatives of several of the quantities occurring, after differentiation, in the right members of these equations in the form which we shall now give. We have

$$\frac{[\rho]}{\rho_1} = \frac{2(u_{1n}^2 - c_1^2)}{2c_1^2 + (\gamma-1)u_{1n}^2}, \tag{49}$$

$$\frac{\rho}{\rho_1} = \frac{(\gamma+1)u_{1n}^2}{2c_1^2 + (\gamma-1)u_{1n}^2}, \tag{50}$$

$$\frac{[\rho]}{\rho} = \frac{2(u_{1n}^2 - c_1^2)}{(\gamma+1)u_{1n}^2}, \tag{51}$$

where c_1, defined by $c_1^2 = \gamma p_1/\rho_1$, is the velocity of sound in the incident flow. These relations follow immediately from the shock conditions in §1. Differentiating with respect to s and performing certain reductions, we find

$$\frac{d}{ds}\left(\frac{[\rho]}{\rho_1}\right) = \frac{d}{ds}\left(\frac{\rho}{\rho_1}\right) = -\frac{4(\gamma+1)c_1^2 u_{1n}u_t\kappa}{[2c_1^2 + (\gamma-1)u_{1n}^2]^2}, \tag{52}$$

$$\frac{d}{ds}\left(\frac{[\rho]}{\rho}\right) = -\frac{4c_1^2 u_t \kappa}{(\gamma+1)u_{1n}^3}. \tag{53}$$

Also differentiation of $c^2 = \gamma p/\rho$ gives

$$\frac{dc^2}{ds} = c^2\left(\frac{1}{p}\frac{dp}{ds} - \frac{1}{\rho}\frac{d\rho}{ds}\right) \tag{54}$$

where dp/ds is given by (16) and $d\rho/ds$ by (52).

Differentiating equations (8), $\cdots$, (10) we have

$$\frac{dA_{11}}{ds} = \frac{[\rho]u_n}{\rho_1}\frac{d\kappa}{ds} + \frac{[\rho]\kappa}{\rho_1}\frac{du_n}{ds} - \frac{4(\gamma+1)c_1^2 u_{1n}u_n u_t \kappa^2}{[2c_1^2 + (\gamma-1)u_{1n}^2]^2}, \tag{55}$$

$$\frac{dA_{12}}{ds} = \frac{4u_n u_t}{\gamma+1}\frac{d\kappa}{ds} + \frac{4u_t\kappa}{\gamma+1}\frac{du_n}{ds} + \frac{4u_{1n}u_n\kappa^2}{\gamma+1}, \tag{56}$$

$$\begin{aligned}\frac{dA_{21}}{ds} = {} & \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{4}{\gamma+1}\right)\left[u_n u_t \frac{d\kappa}{ds} + u_t\kappa\frac{du_n}{ds} + u_{1n}u_n\kappa^2\right] \\ & + \left[\frac{[\rho]}{\rho}\frac{d}{ds}\left(\frac{[\rho]}{\rho_1}\right) + \frac{[\rho]}{\rho_1}\frac{d}{ds}\left(\frac{[\rho]}{\rho}\right)\right]u_n u_t\kappa,\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{dA_{22}}{ds} = {} & \frac{1}{u_n^2 - c^2}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t^2\right\}\left[c^2 u_n\frac{d\kappa}{ds} + c^2\kappa\frac{du_n}{ds} + u_n\kappa\frac{dc^2}{ds}\right] \\ & - \frac{c^2 u_n\kappa}{(u_n^2 - c^2)^2}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t^2\right\}\left[2u_n\frac{du_n}{ds} - \frac{dc^2}{ds}\right] \\ & + \frac{c^2 u_n\kappa}{u_n^2 - c^2}\left\{v^2\frac{d}{ds}\left(\frac{[\rho]}{\rho_1}\right) - \frac{[\rho]}{\rho_1}u_t^2\frac{d}{ds}\left(\frac{[\rho]}{\rho}\right) - \frac{[\rho]}{\rho}u_t^2\frac{d}{ds}\left(\frac{[\rho]}{\rho_1}\right)\right. \\ & \left. + \frac{2A_{21}[\rho]}{\rho_1} - 2\left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t u_{1n}\kappa\right\},\end{aligned} \tag{58}$$

where we have used the substitution $dv^2/ds = 2A_{21}$ which follows from (17) in the derivation of (58). Substitution from equations (44), (52), (53) and (54) for derivatives occurring in the right members of (55), $\cdots$, (58) and substitution from the shock conditions (1), (2) and (3) lead to the explicit formulation of the allowable determinations of the above derivatives of the invariants A_{ij} under the hypothesis of uniform incident flow. In a similar manner we find the allowable determinations of the derivatives of the a_i from the relations

$$\begin{aligned}\frac{da_1}{ds} = {} & \frac{2u_t}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}\frac{d\kappa}{ds} \\ & - \frac{2u_t}{(\gamma+1)u_n^2}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}\kappa\frac{du_n}{ds} \\ & + \frac{2(\gamma-1)u_t\kappa}{(\gamma+1)u_n}\frac{d\rho}{ds} + \frac{2u_{1n}\kappa^2}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\},\end{aligned} \tag{59}$$

$$\frac{da_2}{ds} = -\frac{\rho}{c^2}\frac{dA_{22}}{ds} - \frac{A_{22}}{c^2}\frac{d\rho}{ds} + \frac{\rho A_{22}}{c^4}\frac{dc^2}{ds}, \tag{60}$$

obtained by differentiation of (12) and the relation $a_2 = -\rho A_{22}/c^2$ which follows from (11) and (13). Finally allowable determinations of the tensors P, Q and R for uniform incident flow are given as the result of the above substitutions (52), (55), $\cdots$, (60) together with (47) and (48) in the right members of the following equations

$$P_{\alpha\beta} = \frac{du_{\alpha,\beta}}{ds} = \frac{d}{ds}(A_{ij}\zeta^i_\alpha\zeta^j_\beta),$$

$$Q_\beta = \frac{dp_{,\beta}}{ds} = -\frac{d}{ds}(\rho A_{i2}\zeta^i_\beta),$$

$$R_\beta = \frac{d\rho_{,\beta}}{ds} = \frac{d}{ds}(a_i\zeta^i_\beta),$$

or,

$$P_{\alpha\beta} = \frac{dA_{ij}}{ds}\zeta^i_\alpha\zeta^j_\beta + A_{ij}\frac{d\zeta^i_\alpha}{ds}\zeta^j_\beta + A_{ij}\zeta^i_\alpha\frac{d\zeta^j_\beta}{ds}, \tag{61}$$

$$Q_\beta = -\left[\frac{d\rho}{ds}A_{i2}\zeta^i_\beta + \rho\frac{dA_{i2}}{ds}\zeta^i_\beta + \rho A_{i2}\frac{d\zeta^i_\beta}{ds}\right], \tag{62}$$

$$R_\beta = \frac{da_i}{ds}\zeta^i_\beta + a_i\frac{d\zeta^i_\beta}{ds}. \tag{63}$$

6. Calculation of the coefficient G_1. Consider the equation

$$K = \frac{u_{\alpha,\beta}w^\alpha u^\beta}{v^3}, \tag{64}$$

giving the curvature K of the stream lines where w^α are the components of a vector perpendicular to the stream lines such that $w^1 = -u^2, w^2 = u^1$. Denoting by η the arc length of any stream line measured in the direction of the flow, let us differentiate (64) with respect to η and then evaluate the resulting expression at a point immediately behind the shock line. In this expression and also in all following expressions in this section we shall neglect all terms not involving the derivatives $dK/d\eta$ and $d\kappa/ds$ where κ is the curvature of the shock line and s its arc length; our equations will therefore be valid only in so far as terms involving these derivatives are concerned. We thus obtain

$$\begin{aligned}\frac{dK}{d\eta} &= \frac{u_{\alpha,\beta\gamma}w^\alpha u^\beta u^\gamma}{v^4} = \frac{I_{ijk}\zeta^i_\alpha\zeta^j_\beta\zeta^k_\gamma w^\alpha\xi^\beta_2\xi^\gamma_2}{v^4} \\ &= \frac{I_{i22}(\zeta^i_\alpha w^\alpha)}{v^4} = -\frac{J_{i2}(\zeta^i_\alpha w^\alpha)}{\rho v^4},\end{aligned} \tag{65}$$

where the substitutions (36) and (38) have been made. Now

$$\zeta^1_\alpha w^\alpha = -\frac{e_{\alpha\beta}u^\alpha w^\beta}{u_n} = \frac{-(u_1w_2 - u_2w_1)}{u_n} = \frac{-v^2}{u_n},$$

$$\zeta^2_\alpha w^\alpha = \frac{e_{\alpha\beta}\lambda^\alpha w^\beta}{u_n} = \frac{\lambda_1w_2 - \lambda_2w_1}{u_n} = \frac{u_i}{u_n}.$$

Making these substitutions in the right member of (65) and also substituting for J_{12} and J_{22} from (30) and (42) we find

$$\frac{dK}{d\eta} = \frac{Q_\beta u^\beta}{\rho v^2 u_n} + \frac{u_t}{\rho v^4 u_n}\left(\frac{c^2}{c^2 - u_n^2}\right)(v^2 Q_\beta \lambda^\beta - 2u_t Q_\beta u^\beta). \tag{66}$$

But from (62) we have $Q_\beta = -\rho \zeta_\beta^i \, dA_{i2}/ds$. Hence

$$Q_\beta u^\beta = -\rho \zeta_\beta^i \xi_2^\beta \frac{dA_{i2}}{ds} = -\rho \frac{dA_{22}}{ds},$$

$$Q_\beta \lambda^\beta = -p \zeta_\beta^i \xi_1^\beta \frac{dA_{i2}}{ds} = -\rho \frac{dA_{12}}{ds}.$$

Also from (56) and (58) we have

$$\frac{dA_{12}}{ds} = \frac{4u_n u_t}{\gamma + 1}\frac{d\kappa}{ds},$$

$$\frac{dA_{22}}{ds} = \frac{c^2 u_n}{u_n^2 - c^2}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1 \rho} + \frac{8}{\gamma + 1}\right)u_t^2\right\}\frac{d\kappa}{ds}.$$

Hence,

$$Q_\beta u^\beta = \frac{\rho_1 u_{1n} c^2}{u_n^2 - c^2}\left\{\left(\frac{[\rho]^2}{\rho_1 \rho} + \frac{8}{\gamma + 1}\right)u_t^2 - \frac{v^2[\rho]}{\rho_1}\right\}\frac{d\kappa}{ds}, \tag{67}$$

$$Q_\beta \lambda^\beta = -\frac{4\rho_1 u_{1n} u_t}{\gamma + 1}\frac{d\kappa}{ds}. \tag{68}$$

When we substitute (67) and (68) into the right member of (66) we find that the resulting equation can be given the form

$$\frac{dK}{d\eta} = G_1(M, \alpha)\frac{d\kappa}{ds}, \tag{69}$$

where,

$$G_0(M, \alpha) = -\left[\frac{\left(\frac{\rho}{\rho_1} + \frac{\rho_1}{\rho} + \frac{6 - 2\gamma}{\gamma + 1}\right)\left(\frac{u_t}{v}\right)^3 - \left(\frac{\rho}{\rho_1} - 1\right)\left(\frac{u_t}{v}\right)}{\frac{u_n^2}{c^2} - 1} + \frac{4}{\gamma + 1}\left(\frac{u_t}{v}\right)\right],$$

and,

$$G_1(M, \alpha) = \frac{-G_0(M, \alpha)}{(u_t/v)}\left[1 + \frac{2(u_t/v)^2}{(u_n/c)^2 - 1}\right] - \frac{4}{\gamma + 1}\left[1 + \frac{(u_t/v)^2}{(u_n/c)^2 - 1}\right].$$

The fact that G_0 depends on M and α alone follows from the results of the paper *Calculation of the Curvatures of Attached Shock Waves* (*loc. cit.*) in which each of the ratios occurring in the above expression for G_0 has been calculated explicitly and found to be a function of these variables. It follows from these results and the expression for G_1 that this coefficient likewise depends on M and α alone.

7. Form of the consistency relations. The nature of the coefficients G_0 and G_1 as stated in §2 has now been verified by direct calculation and one can easily establish the form $h(M, \alpha)\kappa^2$ of the polynomial H_1 in equations (19) on the basis of the derivations of §5. In general the remarks of §2 concerning the coefficients G_n and the coefficients of the polynomials H_n in equations (19) can be justified by a simple application of dimensional analysis.

To the above end let us observe that the second derivatives of the quantities u_α, p and ρ were determined essentially as solutions of the linear equations (26), (27), (28) and (32), (33), (34) although the invariants I, J and K were introduced in the course of finding these solutions. Since the components $P_{\alpha\beta}$, Q_β and R_β were determined by direct differentiation of the right members of (7) along the shock line it is seen that these components are homogeneous and linear in $d\kappa/ds$ and κ^2. Hence the second derivatives of u_α, p and ρ will be homogeneous and linear in $d\kappa/ds$ and κ^2 in view of their determination as solutions of the above linear system and the fact that the first derivatives of u_α, p and ρ are proportional to κ. Differentiation of (64) with respect to η and substitution of the above expressions for the second derivatives of the u_α as well as corresponding expressions for first derivatives of u_α leads to an expression for $dK/d\eta$ which is homogeneous and linear in $d\kappa/ds$ and κ^2. Now each of the quantities $dK/d\eta$, $d\kappa/ds$ and κ^2 has dimension L^{-2}. Since each of the terms in the equation for $dK/d\eta$ must have the same dimension it follows that the coefficients of the terms in $d\kappa/ds$ and κ^2 in this equation are dimensionless quantities.

The above procedures and remarks, applicable to equation (19) for $n = 1$, extend to the general case. Thus there exist relations of the form (19) with $n = 0, 1, 2 \cdots$ such that each term in the general equation (19) has dimension $1/L^{n+1}$ which is the dimension of the left member $K_{(n)}$. Since $\kappa_{(n)}$ has dimension $1/L^{n+1}$ it follows that its coefficient G_n is dimensionless. Moreover it is easily seen from the derivation that each term in the polynomial H_n is of dimension $1/L^{n+1}$ in the product of κ and its derivatives with respect to s (these derivatives of κ being of order less than n). Hence the coefficients in the polynomial H_n have zero dimension.

Now it is easily seen that the coefficient G_n and the above coefficients of the polynomial H_n are expressible, on account of the conditions (1), (2) and (3), as functions of w, p_1, ρ_1 and α where w is the magnitude of the velocity before the shock line, p_1 and ρ_1 the pressure and density before the shock, and α is the inclination of the shock line. In fact, it is clear that each such quantity is a rational function $R(w, p_1, \rho_1, \alpha)$ of w, p_1 and ρ_1 with coefficients depending on the inclination angle α. Now $M^2 = w^2/c_1^2$ or $M^2 = w^2\rho_1/\gamma p_1$. Hence $\rho_1 = \gamma M^2 p_1/w^2$ and making this substitution the above function R becomes $r(w, p_1, M^2, \alpha)$. But w has dimension L/T and p_1 has dimension m/LT^2 where we have used m to represent mass rather than M (mach number of uniform incident flow). Now M and α are dimensionless quantities. Hence if $r(w, p_1, M^2, \alpha)$ actually depended on the pressure p_1 it could not be of zero dimension. Removing p_1 as one of the possible variables on which r depends we see that r must similarly be independent of w.

The above result can now be stated as follows. *For uniform flow before the shock line there exist relations of the form* (19) *with* $n = 0, 1, 2, \cdots$ *in which the coefficients* G_n *and the coefficients of the polynomials* H_n *are rational functions of* M^2 *with coefficients depending on the inclination* α *of the shock line.*

8. Tables and graphs. In the paper *Calculation of the Curvatures of Attached Shock Waves* we constructed the graphs of the function $-G_0(M, \alpha)$ for selected values of M; the discussion of the tables used in the construction of these graphs likewise applies here and will not be repeated. The following Tables I to XIII are based on the value $\gamma = 1.405$ and involve the values of M previously used in constructing the graphs of the function $-G_0(M, \alpha)$. Graphs of the function $G_1(M, \alpha)$ are constructed from the data of these tables. It is seen that each of these graphs crosses the α-axis two times; we denote, corresponding to any value of M, the singular values of α, i.e. those values of α for which $G_1(M, \alpha) = 0$, by $\alpha_1(M)$ and $\alpha_2(M)$ in increasing order of magnitude (see §3). The values of $\alpha_1(M)$ and $\alpha_2(M)$ have been read from the above graphs of the function $G_1(M, \alpha)$ and entered in Table XIV. Finally the graphs of the functions $\alpha(M)$ have been constructed.

TABLE I

M = 1.05 α_0(M) = 72.250°

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
73	829.9979	77	2.8127	80	−.8308	88	.7969
73.5	207.9565	78	+.2527	81	−.7119	90	.9158
74	90.1773	79	−.5220	82	−.5299		
75	25.6800	79.5	−.7771	84	+.0158		
76	8.4837	79.937	−.8282	86	.4966		

TABLE II

M = 1.08 α_0(M) = 67.8°

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
70	93.7283	76	−.3265	80	−.5847	88	.8782
72	14.2347	77.558	−.8226	82	−.1453	90	.9663
74	+2.2702	78	−.8319	84	+.2985		
75	+.4850	79	−.7523	86	.6611		

TABLE III

M = 1.12 α_0(M) = 63.238°

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
64	161.6594	73	−.1420	79	−.4769	90	1.0347
67	34.6556	75.207	−.8116	82	+.1165		
70	+4.1668	76	−.8328	85	.6510		
72	+.6652	77	−.7727	88	.9690		

TABLE IV

$M = 1.18 \quad \alpha_0(M) = 57.944°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
60	218.7093	70	−.0578	72	−.7057	84	+.6502
61.5	62.7965	70.1	−.1018	72.653	−.7889	87	+1.0033
63	25.4481	70.2	−.1625	74	−.8359	89.4	+1.1516
64	15.1079	70.3	−.2050	75	−.7892	89.7	+1.0899
66	5.3883	70.4	−.2471	78	−.4022	90	1.1375
69	+.5982	70.5	−.2909	80.2	−.0081		
69.5	+.2180	71	−.4667	81	+.1405		

TABLE V

$M = 1.25 \quad \alpha_0(M) = 53.170°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
54	1571.23	60	16.0284	70.534	−.7505	80	+.1037
56	131.8270	62	7.1397	72	−.8414	82	.4603
57	67.4703	66	+.9988	74	−.7737	86	1.0409
58	39.4192	68	−.1725	76	−.5573	90	1.2590
59	24.6109	70	−.6806	78	−.2495		

TABLE VI

$M = 1.35 \quad \alpha_0(M) = 47.8°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
48	+31487.0000	64	.6153	72	−.8407	88	1.3841
52	69.4521	66	−.1775	74	−.7088	90	1.4324
54	28.7633	68	−.6238	76	−.4674		
56	14.4170	68.563	−.7024	80	+.2160		
60	4.0413	71	−.8540	84	+.9233		

TABLE VII

$M = 1.47 \quad \alpha_0(M) = 42.869°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
45	299.0665	57	4.0892	66.870	−.6082	85	1.2283
46	139.8144	60	1.6882	70	−.8633	90	1.6389
50	25.1208	62	+.6940	72.5	−.8170		
53	11.0656	64	−.0020	75	−.5831		
55	6.7188	65	−.2620	80	+.2768		

TABLE VIII

M = 1.63 $\alpha_0(M) = 37.843°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
40	257.8569	60	.8953	67.5	−.7361	80	+.2898
45	27.5047	62	+.2709	70	−.8749	85	1.4022
50	8.8818	65	−.3961	72.5	−.8510	90	1.9088
55	3.3945	65.635	−.5003	75	−.6269		

TABLE IX

M = 1.83 $\alpha_0(M) = 33.120°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
35	348.4129	60	.6324	72.5	−.9120	90	2.2313
40	29.8013	62	.1577	75	−.7221		
45	10.5435	65	−.4039	78	−.2269		
50	4.9257	67.5	−.7289	80	+.2416		
55	2.2470	70	−.9068	85	1.5727		

TABLE X

M = 2.12 $\alpha_0(M) = 28.144°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
30	302.3233	55	1.7244	72.5	−.9840	85	1.7566
35	27.4975	60	+.5658	75	−.8665	90	2.6609
40	10.6598	64.582	−.2669	77.5	−.5068		
45	5.6698	65	−.3311	78	−.4031		
50	3.2562	70	−.9027	80	+.1046		

TABLE XI

M = 2.56 $\alpha_0(M) = 22.994°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
25	203.9555	55	1.4647	72.5	−1.0276	85	+1.9078
30	22.0977	60	+.6075	75	−1.0268	90	+3.2180
35	16.2187	65	−.1932	77.5	−.7565		
45	3.6435	70	−.8458	80	−.1477		

TABLE XII

M = 3.24 $\alpha_0(M) = 17.150°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
20	138.2446	55	1.3235	72.5	−1.0039	85	1.9736
25	16.6743	60	+.6756	75	−1.1400	87.5	3.3077
30	7.6440	65	−.02653	77.5	−1.0133	90	3.8669
40	3.4966	65.399	−.08338	80	−.4824		
50	1.9476	70	−.7268	82.5	+.5496		

TABLE XIII

M = 4.45 $\alpha_0(M) = 12.982°$

α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1	α(deg.)	G_1
15	83.0698	50	1.6516	70	−.5715	82	−.0732
20	11.3827	55	1.2283	72.5	−.9111	85	+1.9096
25	5.6481	60	.7341	75	−1.1721	87.5	3.7511
35	2.9259	65	+.1308	77.5	−1.2139	89	4.4479
45	1.9945	66.246	−.03721	80	−.8359	90	4.5847

TABLE XIV

M	α_1(deg.)	α_2(deg.)	M	α_1(deg.)	α_2(deg.)
1.05	78.23	83.95	1.63	62.95	78.90
1.08	75.50	82.65	1.83	62.70	79.10
1.12	72.78	81.52	2.12	62.90	79.50
1.18	69.90	80.25	2.56	63.70	80.55
1.25	67.60	79.50	3.24	64.70	81.50
1.35	65.23	78.79	4.45	65.90	82.0
1.47	64.00	78.65			

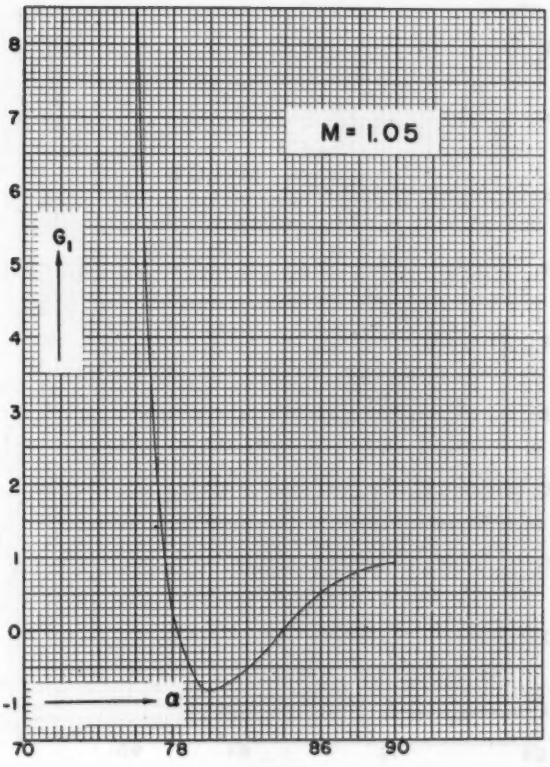

FIG. 2

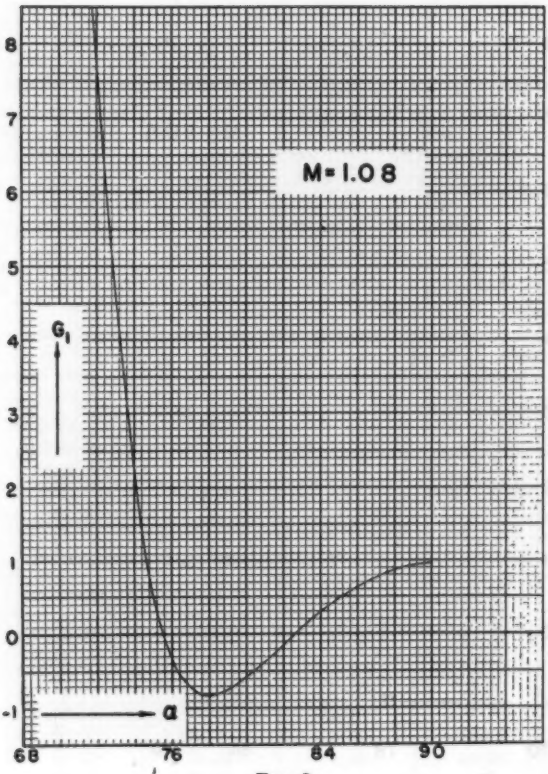

FIG. 3

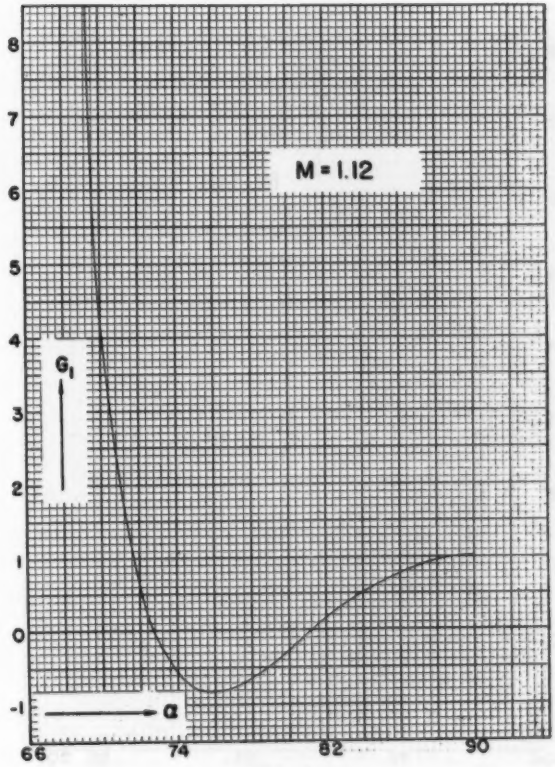

FIG. 4

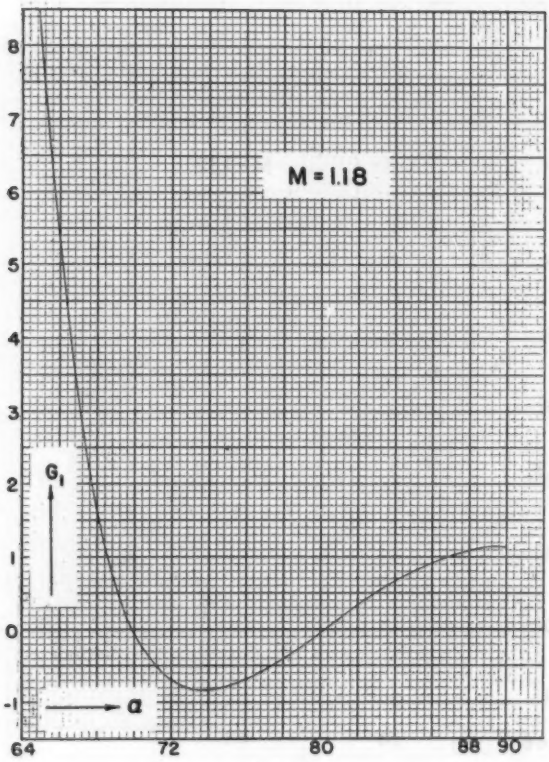

FIG. 5

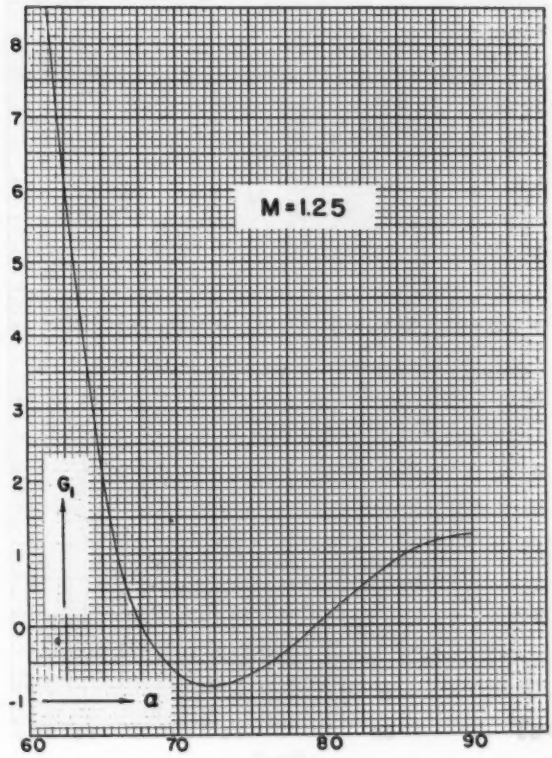

FIG. 6

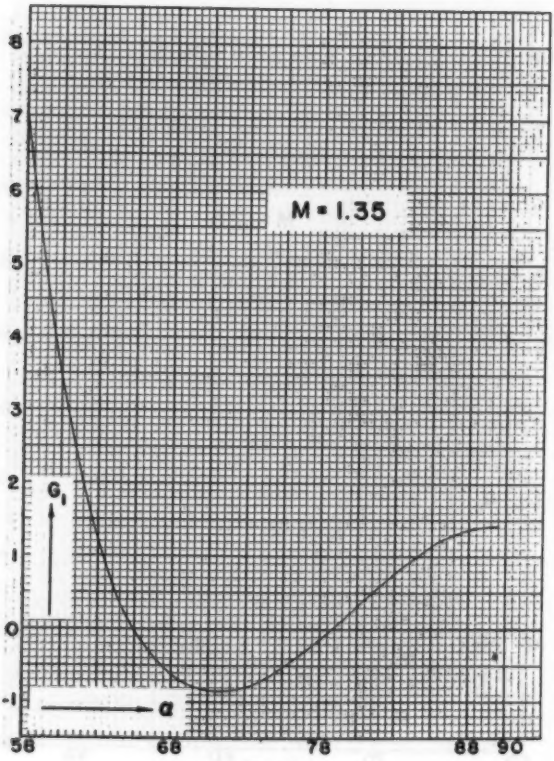

FIG. 7

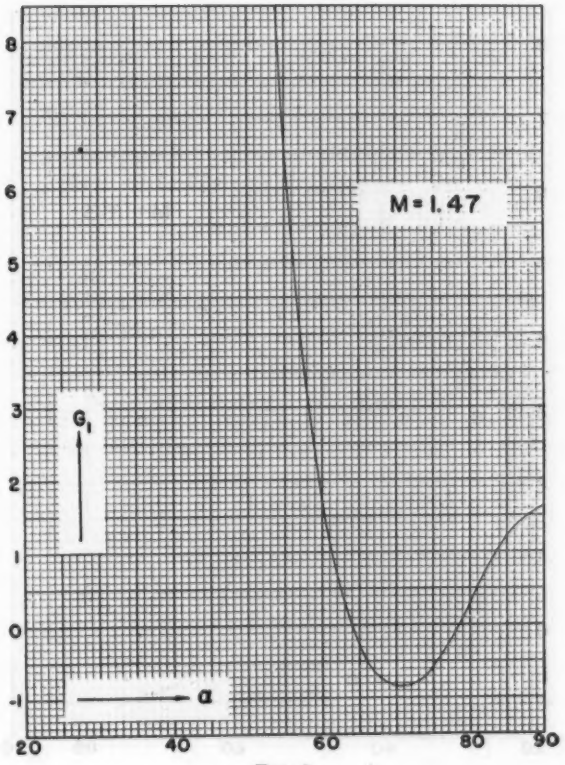

FIG. 8

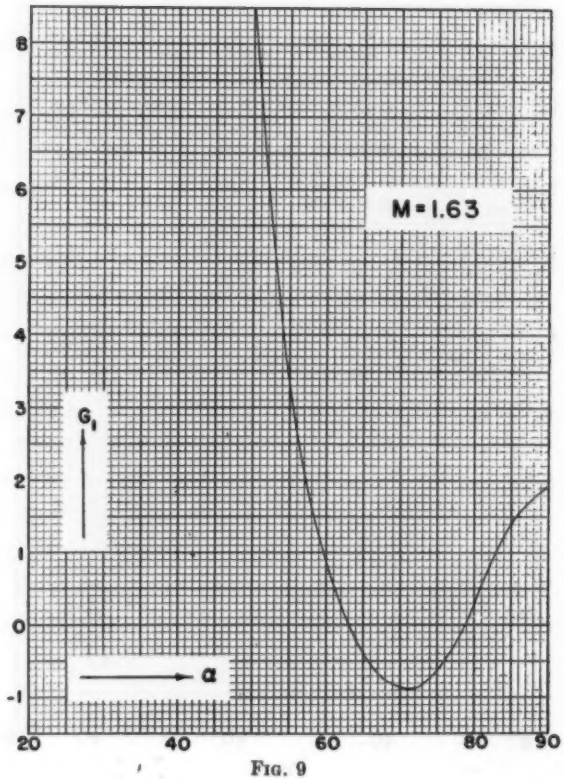

FIG. 9

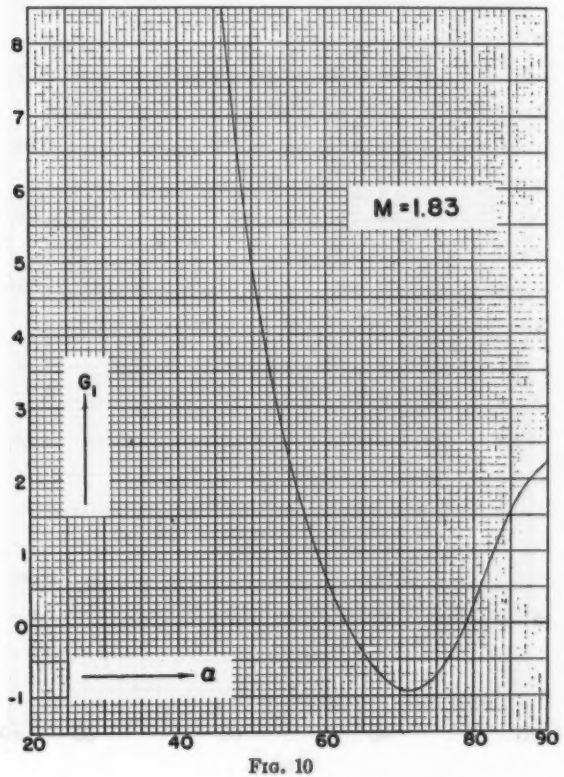

FIG. 10

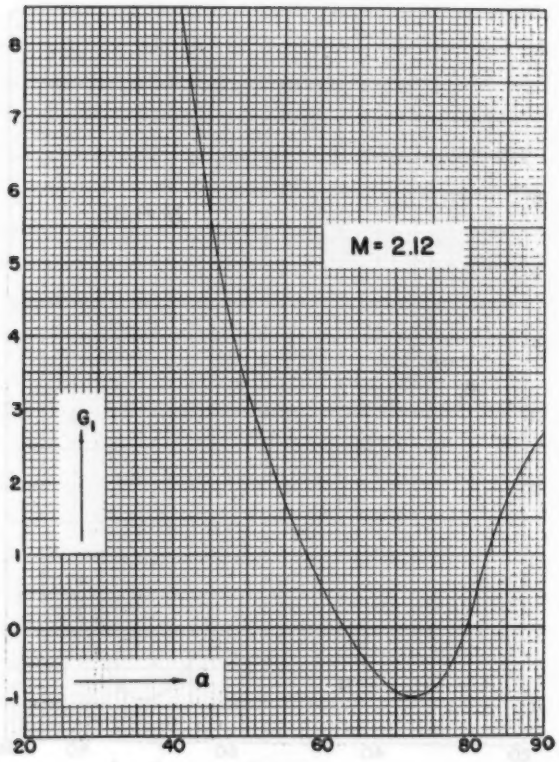

FIG. 11

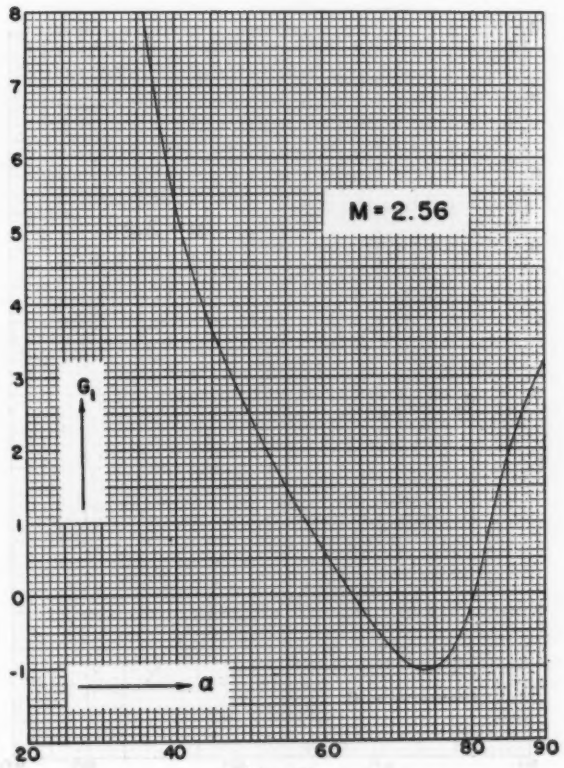

FIG. 12

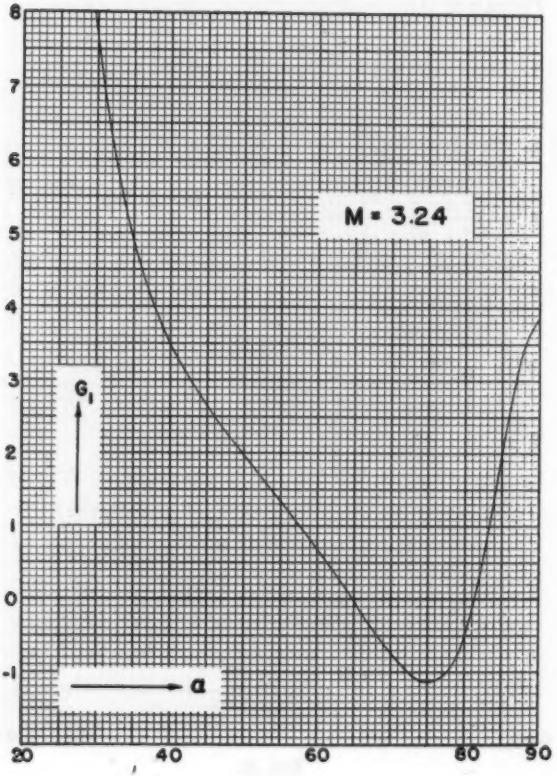

FIG. 13

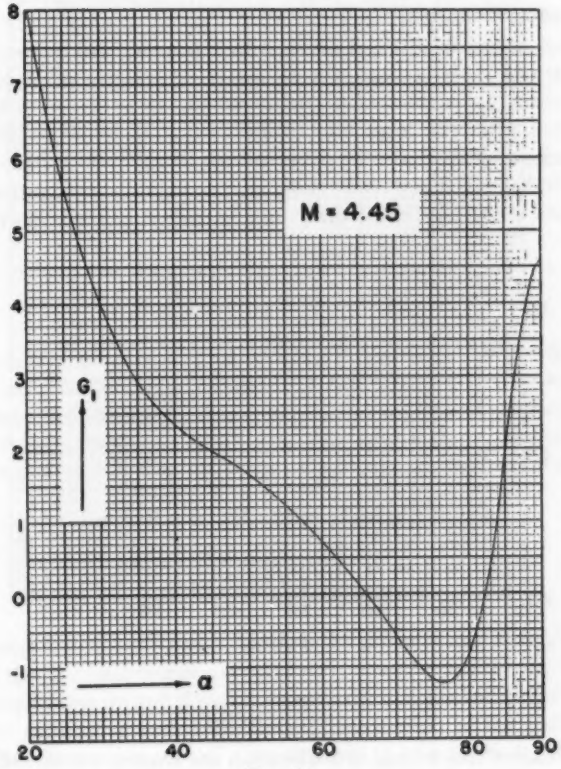

FIG. 14

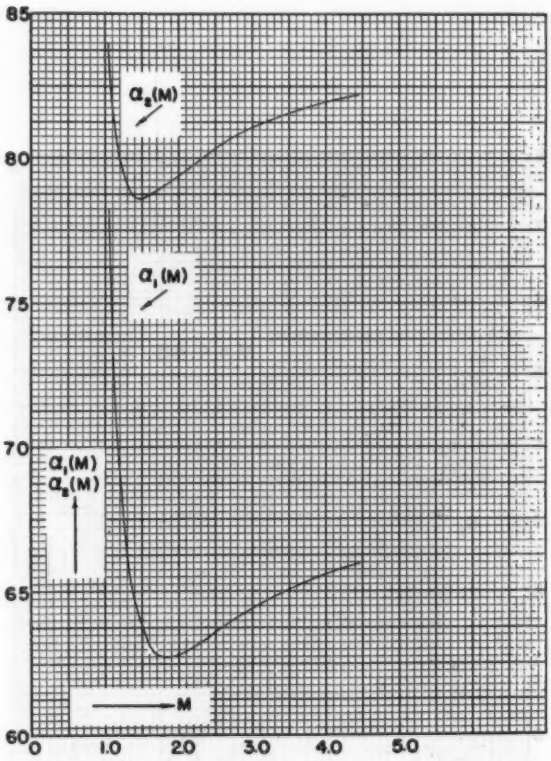

FIG. 15

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2. *Calculation of the Curvatures of Attached Shock Waves*, Journal of Mathematics and Physics, vol. 28, 1949, page 99. Since writing this paper, the attention of the author has been directed to the fact that there are certain points of similarity between it and the article by L. Crocco, *Singolarità della corrente gassosa iperacustica nell' interno di una prora a diedro*, Atti del I° Congresso dell' Unione Matematica Italiana, vol. 15, 1937, pp. 597–615.

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(Received June 30, 1948)

Reprinted from JOURNAL OF MATHEMATICS AND PHYSICS
Vol. XXVIII, No. 2, July, 1949

ON CONDITIONS FOR STEADY PLANE FLOW WITH SHOCK WAVES*

BY T. Y. THOMAS

1. Introduction. Consider the plane flow of a gas in which there is present a shock wave or shock line $\Sigma(t)$. We refer the motion to rectangular coordinates x^α ($\alpha = 1, 2$) and assume that the curve $\Sigma(t)$ is represented by an equation $F(x, t) = 0$ where t is the time and the function $F(x, t)$ is analytic in the variables x^α and t and such that $F_{,\sigma}\, F_{,\sigma} > 0$ (condition of regularity). Denote by ν_α the components of the unit normal vector to $\Sigma(t)$ at any time t, this vector ν being directed into the region of flow behind the shock line; also let G denote the velocity of the wave in the direction of the normal ν. Then if we choose the algebraic sign of the function $F(x, t)$ so that $F(x, t) < 0$ in the region of flow before the shock line at any time t, and $F(x, t) > 0$ in the region behind the shock line, we shall have

$$(1) \qquad \nu_\alpha(x, t) = \frac{F_{,\alpha}}{\sqrt{F_{,\sigma}\, F_{,\sigma}}}, \qquad G(x, t) = -\frac{\partial F/\partial t}{\sqrt{F_{,\sigma}\, F_{,\sigma}}}.$$

We assume in the following discussion that the flow before the shock line (incident flow) is uniform, i.e. the velocity, pressure and density are constant. The flow behind the shock line may be variable, non-adiabatic and rotational but the viscosity and thermal conductivity will be assumed to be zero. Hence the differential equations governing the flow behind the shock line will be given by

$$(2) \qquad p_{,\alpha} + \rho(\partial u_\alpha/\partial t) + \rho u_\sigma u_{\alpha,\sigma} = 0,$$

$$(3) \qquad (\partial \rho/\partial t) + \rho_{,\alpha} u_\alpha + \rho u_{\alpha,\alpha} = 0,$$

$$(4) \qquad (\partial p/\partial t) - \rho u_\sigma(\partial u_\sigma/\partial t) - \rho u_{\alpha,\beta}\, u_\alpha u_\beta + \gamma p u_{\sigma,\sigma} = 0,$$

where u_α denotes the velocity components, p the pressure and ρ the density; the quantity γ is the thermodynamic constant c_p/c_v where c_p and c_v are the specific heats at constant pressure and density respectively. Equation (4) expresses the condition that entropy is constant following the motion of the individual gas particles. The functions u_α, p and ρ are assumed to be analytic functions of the variables x^α, t in the region behind the shock line $\Sigma(t)$ and also along this line at any time t.

It will be supposed specifically that the shock wave $\Sigma(t)$ originates at the vertex V of a pointed obstacle in the uniform flow and that the wave passes through V at all times t. This requirement has its expression in the condition $F(0, t) = 0$ independently of t. In §3 we have stated conditions under which the shock line $\Sigma(t)$ will be fixed in the plane of the flow and the flow behind the shock line will be steady in character. The proof of this theorem is contained in the subsequent paragraphs.

* Prepared under Navy Contract N6onr-180, Task Order V, with Indiana University.

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2. Initial direction of shock line. The shock conditions for the type of flow described in §1 can be represented by equations of the form [1]

$$[u_n] = \frac{2\{\rho_1(G - u_{1n})^2 - \gamma p_1\}\, \nu_n}{(\gamma + 1)\rho_1(G - u_{1n})}, \tag{5}$$

$$[p] = \frac{2\{\rho_1(G - u_{1n})^2 - \gamma p_1\}}{\gamma + 1}, \tag{6}$$

$$[\rho] = \frac{2\rho_1\{\rho_1(G - u_{1n})^2 - \gamma p_1\}}{2\gamma p_1 + (\gamma - 1)\rho_1(G - u_{1n})^2}, \tag{7}$$

where the bracket denotes, as usual, the difference in the values of the quantity enclosed on the two sides of the shock line; also the subscript is used on quantities when considered in the region before the shock line. The quantity u_{1n} in the above equations denotes the component of the incident velocity along the normal ν to the shock line.

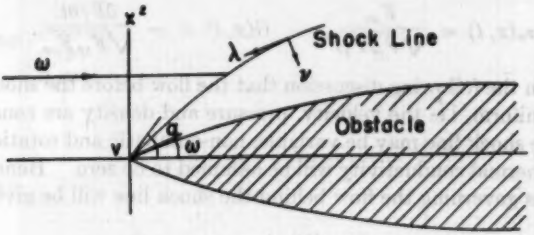

FIG. 1

We suppose that the vertex V of the obstacle is located at the origin of coordinates and that the direction of the incident flow is that of the positive direction of the x^1 axis. Denote by α the inclination of the shock line and by ω the inclination of the obstacle at V. Then it follows readily from the two equations (5) that

$$\tan \omega = \frac{-2\{(G/c_1 - M \sin \alpha)^2 - 1\} \cos \alpha}{2(G/c_1 - M \sin \alpha)^2 \sin \alpha + (\gamma + 1)M(G/c_1 - M \sin \alpha) - 2 \sin \alpha}, \tag{8}$$

where $c_1^2 = \gamma p_1/\rho_1$ is the square of the velocity of sound in the uniform incident flow and M its mach number. But from $F(0, t) = 0$ and the second equation (1) we have $G(0, t) = 0$ for all t. Hence at the point V equation (8) reduces to the relation between the angles α and ω which holds under the hypothesis of steady flow. *It follows that the angle α will have a fixed value independent of the time t and that its determination will be the same as for the case of steady flow.* The requirement $[\rho] > 0$ (condition for compression shock) combined with the condition (8) leads to the inequality $\alpha_0(M) < \alpha \leqq 90°$ where $\alpha_0(M)$ is the value of the angle arc sin M in the first quadrant [2].

3. Singular shock directions and theorem. Suppose for the moment that the flow is steady with shock line Σ issuing from the vertex V of the obstacle (as

shown in the figure). Let κ denote the curvature of the shock line and K the curvature along any stream line in the region behind the shock line. Let s be the arc length along the shock line so that $dx^\alpha/ds = \lambda^\alpha$ where λ is the unit tangent vector to the shock line (see figure) and η the arc length along one of the above stream lines measured in the direction of the flow. Put $K_{(n)} = d^nK/d\eta^n$ and $\kappa_{(n)} = d^n\kappa/ds^n$ for $n = 0, 1, 2, \cdots$ so that $K_{(0)} = K$ and $\kappa_{(0)} = \kappa$. It has then been shown for the case now under consideration (steady flow behind the shock line) that there exist relations of the form [3]

$$K_{(n)} = G_n(M, \alpha)\kappa_{(n)} + H_n, \qquad n = 0, 1, 2, \cdots, \tag{9}$$

at point V, where the H_n are polynomials in κ and its derivatives with respect to s of order less than n; moreover, the quantities $G_n(M, \alpha)$ and the coefficients of the polynomials H_n are rational functions of M^2 with coefficients depending on the inclination α of the shock line. For a given value of $M > 1$, a value of α for which any one of the functions $G_n(M, \alpha) = 0$ is called *singular relative to* M.

In the case of variable flow behind the shock line any value of α satisfying an equation $G_n(M, \alpha) = 0$ will likewise be called singular relative to M. This is permissible since we have shown that the determination of α is the same as for steady flow and that its value is independent of the time t. There will be at most a countable infinity of singular values of α relative to a given value of M. It is the object of this paper to prove the following main result. *The shock line $\Sigma(t)$ will be fixed in the plane of the flow and the flow behind the shock line will be steady provided that the inclination α of the shock line is non-singular relative to the mach number M of the uniform incident flow.*

4. Higher derivatives behind the shock line. Put $t = x^3$ so that in addition to the two space coordinates x^1, x^2 we now have the time variable x^3. We denote the set of these three variables by x^i $(i = 1, 2, 3)$; in this connection we shall use Greek indices when the range 1, 2 is intended and Latin indices for the range 1, 2, 3. Now consider a point with coordinates x^α on the shock line at time t which can be denoted merely by the letter x^i. Then the displacement dx^i produces changes in the velocity components u_α on the two sides of the shock line given in general by

$$u_{1\alpha} + u_{1\alpha,i}\,dx^i + (1/2!)u_{1\alpha,ij}\,dx^i dx^j + \cdots,$$

and,

$$u_{2\alpha} + u_{2\alpha,i}\,dx^i + (1/2!)u_{2\alpha,ij}\,dx^i dx^j + \cdots,$$

where the subscript 1 refers to quantities before the shock line (region 1) and the subscript 2 to quantities behind this line (region 2). For the case of uniform flow in region 1, with which we deal, the coefficients of dx^i and higher powers of dx^i vanish in the first of the above series. When we impose the condition that $x^\alpha + dx^\alpha$ represents a point on the shock line (at time $t + dt$) we can write

$$[u_\alpha + u_{\alpha,i}\,dx^i + (1/2!)u_{\alpha,ij}\,dx^i dx^j + \cdots] = [u_\alpha] \text{ at } x^i + dx^i, \tag{10}$$

i.e. these relations hold under the condition

$$F_{,i}\,dx^i + (1/2!)F_{,ij}\,dx^i dx^j + \cdots = 0. \tag{11}$$

Now $u_{1n} = u_{1\alpha}\nu^\alpha = w\nu^1$ in equations (5), (6) and (7) where w is the magnitude of the velocity of the incident flow. Substituting for ν^1 from (1) the quantity u_{1n} becomes an expression in the spatial derivatives of the function F. When we substitute this expression for u_{1n} into the right members of (5), (6) and (7), and also the corresponding expressions for ν_α and G given by (1), these conditions become

$$[u_\alpha] = A_\alpha, \qquad [p] = B, \qquad [\rho] = C \tag{12}$$

where A_α, B and C depend on derivatives of F and hence can be thought of as functions of the variables x^i. Using the first equation (12) the right member of (10) becomes

$$A_\alpha + A_{\alpha,i}\,dx^i + (1/2!)A_{\alpha,ij}\,dx^i dx^j + \cdots,$$

where the coefficients A_α, $A_{\alpha,i}$, $\cdots$ in this series are evaluated at x^i. When this substitution is made in (10) and all terms moved to the left member we have

$$\begin{aligned}(u_{\alpha,i} - A_{\alpha,i})dx^i + (1/2!)(u_{\alpha,ij} - A_{\alpha,ij})dx^i dx^j + \cdots \\ + (1/m!)(u_{\alpha,ij\ldots k} - A_{\alpha,ij\ldots k})dx^i dx^j \cdots dx^k + \cdots = 0,\end{aligned} \tag{13}$$

where the derivatives of the u_α which appear in this equation are evaluated on the side of the shock line facing the region 2; derivatives of the $u_{1\alpha}$, p_1 and ρ_1 do not appear since these quantities are assumed to be constant in the region 1. The equation (13) is satisfied whenever (11) is satisfied. Similarly we see that whenever (11) holds we have

$$(p_{,i} - B_{,i})dx^i + (1/2!)(p_{,ij} - B_{,ij})dx^i dx^j + \cdots = 0, \tag{14}$$

$$(\rho_{,i} - C_{,i})dx^i + (1/2!)(\rho_{,ij} - C_{,ij})dx^i dx^j + \cdots = 0. \tag{15}$$

5. A lemma. Consider the following two equations

$$g_{\alpha\ldots\beta}x^\alpha \cdots x^\beta + g_{\alpha\ldots\beta\gamma}x^\alpha \cdots x^\beta x^\gamma + \cdots = 0, \tag{16}$$

$$h_\alpha x^\alpha + h_{\alpha\beta}x^\alpha x^\beta + h_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma + \cdots = 0, \tag{17}$$

in which the left members are convergent series in the neighborhood of $x^\alpha = 0$ and $h_\alpha h_\alpha > 0$, i.e. not all the coefficients h_α vanish. *We suppose in this section that indices may have the values* $1, \cdots, n$ *and that repeated indices are summed over this range.* Equation (17) defines a surface S_{n-1} passing through $x^\alpha = 0$ and regular in the neighborhood of the origin. We prove the following

Lemma. If (17) *implies* (16) *then* $g_{\alpha\ldots\beta}x^\alpha \cdots x^\beta = 0$ *is satisfied whenever* $h_\alpha x^\alpha = 0$ *is satisfied.*

Now the h_α are the components of a normal vector to the surface (17) at $x^\alpha = 0$. Let λ^α define a tangent vector to S_{n-1} at $x^\alpha = 0$. We may take λ to be a unit vector without loss of generality (euclidean definition of distance). Then $h_\alpha\lambda^\alpha = 0$; conversely it follows from this equation that the λ^α are the components of a tangent vector to the surface S_{n-1} at the origin. The plane determined by λ and

the normal vector at $x^\alpha = 0$ will intersect the surface S_{n-1} in an analytic curve through the origin. Let $x^\alpha(s)$, with s the arc length measured from the origin, denote this curve. Then $x^\alpha(s)/s \to \lambda^\alpha$ as $s \to 0$. Now (16) and (17) are satisfied by $x^\alpha = x^\alpha(s)$. Making this substitution and dividing by s we have

$$g_{\alpha\cdots\beta}\left(\frac{x^\alpha}{s}\right)\cdots\left(\frac{x^\beta}{s}\right) + g_{\alpha\cdots\beta\gamma}\left(\frac{x^\alpha}{s}\right)\cdots\left(\frac{x^\beta}{s}\right)x^\gamma + \cdots = 0,$$

$$h_\alpha\left(\frac{x^\alpha}{s}\right) + h_{\alpha\beta}\left(\frac{x^\alpha}{s}\right)x^\beta + h_{\alpha\beta\gamma}\left(\frac{x^\alpha}{s}\right)x^\beta x^\gamma + \cdots = 0.$$

Then allowing s to approach zero these equations become $g_{\alpha\cdots\beta}\lambda^\alpha\cdots\lambda^\beta = 0$ and $h_\alpha\lambda^\alpha = 0$. Hence $g_{\alpha\cdots\beta}\lambda^\alpha\cdots\lambda^\beta = 0$ whenever $h_\alpha\lambda^\alpha = 0$ and the lemma is proved.

6. Determination of higher derivatives. The above lemma has direct application in the determination of the first and higher derivatives of the quantities u_α, p and ρ immediately behind the shock line.[1] From (11) and (13) we have $(u_{\alpha,i} - A_{\alpha,i})dx^i = 0$ whenever $F_{,i}\,dx^i = 0$. It follows that the quantities $u_{\alpha,i} - A_{\alpha,i}$ are proportional, for α fixed, to the $F_{,i}$. Hence $u_{\alpha,i} - A_{\alpha,i} = a_\alpha F_{,i}$ where the a_α are the components of a vector defined along the shock line. Or

$$u_{\alpha,\beta} - A_{\alpha,\beta} = a_\alpha F_{,\beta} = \bar{a}_\alpha \nu_\beta, \tag{18}$$

$$(\partial u_\alpha/\partial t) - (\partial A_\alpha/\partial t) = a_\alpha(\partial F/\partial t) = -\bar{a}_\alpha G, \tag{19}$$

when use is made of (1). Similarly from (14) and (15) we have

$$p_{,\alpha} - B_{,\alpha} = bF_{,\alpha} = \bar{b}\nu_\alpha, \tag{20}$$

$$(\partial p/\partial t) - (\partial B/\partial t) = b(\partial F/\partial t) = -\bar{b}G, \tag{21}$$

and

$$\rho_{,\alpha} - C_{,\alpha} = cF_{,\alpha} = \bar{c}\nu_\alpha, \tag{22}$$

$$(\partial\rho/\partial t) - (\partial C/\partial t) = c(\partial F/\partial t) = -\bar{c}G. \tag{23}$$

When we eliminate the derivatives appearing in (2), (3) and (4) by means of (18), $\cdots$, (23) the result is a system of equations in the quantities $\bar{a}_\alpha$, $\bar{b}$ and $\bar{c}$. The solution of this system can easily be found. When the values of the $\bar{a}_\alpha$, $\bar{b}$ and $\bar{c}$ so obtained are inserted into the equations (18), $\cdots$, (23) we arrive at the determination of all first derivatives of the functions u_α, p and ρ at points immediately behind the shock line (at any time t) in terms of derivatives of the function F. We omit the details of the procedure since such is not necessary for our purpose.

The above process can be extended to the determination of derivatives of any

[1] It is supposed in this section, following the form in which conditions (13), (14) and (15) are stated, that the quantities u_α, p and ρ are constant before the shock line. However the process used for the determination of derivatives applies generally subject only to required conditions of differentiability.

order behind the moving shock line $\Sigma(t)$. Thus in the case of second derivatives of the components u_α we multiply (11) by a_α and subtract from (13). On account of (18) and (19) this gives

$$(1/2!)(u_{\alpha,ij} - A_{\alpha,ij} - a_\alpha F_{,ij})dx^i dx^j + \cdots = 0,$$

and this equation holds whenever (11) is satisfied. Applying the lemma we therefore have

$$(u_{\alpha,ij} - A_{\alpha,ij} - a_\alpha F_{,ij})dx^i dx^j = 0$$

whenever $F_{,i}\,dx^i = 0$. Hence

$$u_{\alpha,ij} - A_{\alpha,ij} - a_\alpha F_{,ij} = b_\alpha F_{,i} F_{,j}, \tag{24}$$

for some vector with components b_α along the shock line $\Sigma(t)$. Expanding (24) we obtain

$$u_{\alpha,\beta\gamma} - A_{\alpha,\beta\gamma} - a_\alpha F_{,\beta\gamma} = b_\alpha F_{,\beta} F_{,\gamma} = \bar{b}_\alpha \nu_\beta \nu_\gamma, \tag{25}$$

$$\frac{\partial^2 u_\alpha}{\partial x^\beta \partial t} - \frac{\partial^2 A_\alpha}{\partial x^\beta \partial t} - a_\alpha \frac{\partial^2 F}{\partial x^\beta \partial t} = b_\alpha F_{,\beta} \frac{\partial F}{\partial t} = -\bar{b}\alpha\nu_\beta G, \tag{26}$$

$$\frac{\partial^2 u_\alpha}{\partial t^2} - \frac{\partial^2 A_\alpha}{\partial t^2} - a_\alpha \frac{\partial^2 F}{\partial t^2} = b_\alpha \frac{\partial F}{\partial t} \frac{\partial F}{\partial t} = \bar{b}_\alpha G^2. \tag{27}$$

Similar equations hold for the second derivatives of the functions p and ρ. When these equations are used in conjunction with the equations obtained by differentiation of (2), (3), (4) we obtain the final determination of all second derivatives of u_α, p and ρ in terms of derivatives of the function F at points immediately behind the shock line $\Sigma(t)$. It is evident that this process can be applied successively to the determination of derivatives of higher order.

7. Value of the functions at the vertex V. We may represent the shock line $\Sigma(t)$ by $y = f(x, t)$ where f is an analytic function of x, t and the variables x, y are identified with the previous coordinates x^1, x^2 respectively; then $f(0, t) = 0$ by our hypothesis (§1). To secure the relationship between this representation of the shock line and the previous one by means of the function $F(x, t)$, where x here stands for the x^1, x^2 coordinates, we suppose that $F(x, t) \equiv y - f(x, t)$. But $\alpha = \partial f(0, t)/\partial x$ and we have seen in §2 that α is a constant independent of the time t. Hence the derivatives $\partial F(0, t)/\partial x^\alpha$ are constants independent of t and hence $\partial^2 F(0, t)/\partial x^\alpha \partial t = 0$ for all t. In fact we can write $\partial F(0, t)/\partial x^i = \text{const.}$ and $\partial^2 F(0, t)/\partial x^i \partial t = 0$ using the notation in §4.

Since ν_α are constants and $G = 0$ at the vertex V of the obstacle, independently of t, it follows from (5) and the assumption of uniform incident flow that $u_\alpha(0, t) = \text{const.}$ Similarly from (6) and (7) we have $p(0, t) = \text{const.}$ and $\rho(0, t) = \text{const.}$ with these constants independent of the time t. The quantities u_α, p and ρ have the same algebraic determination at V by means of equations (5), (6) and (7) as in the case of steady flow.

8. Derivatives of the functions at V. Since $\partial u_\alpha/\partial t = 0, \partial p/\partial t = 0$ and $\partial \rho/\partial t = 0$ at V by the result of §7 it follows that the differential equations (2), (3) and (4) express, at V, the same algebraic conditions on the quantities involved as in the case of steady flow. Also from $\partial F(0, t)/\partial t = 0$ and $\partial^2 F(0, t)/\partial x^i \partial t = 0$ at V we see that $\partial G/\partial t = 0$ and $\partial G/\partial x^\alpha = 0$ at V where G is defined by (1). Since also $G = 0$ at V it follows that equations (18), (20) and (22) likewise express the same algebraic conditions on derivatives as are given by these equations for steady flow. Equations (19), (21) and (23) are satisfied automatically, at V, since $G = 0$, $\partial u_\alpha/\partial t = 0$, etc., at this point. Hence we can deduce, at V, the validity of the equation (9), for $n = 0$, for the variable flow behind the shock line since the derivation involves only quantities whose determination is the same as for steady flow [3]. Specifically this equation is $K = G_0(M, \alpha)\kappa$. Assuming α non-singular so that $G_0(M, \alpha) \neq 0$ this equation determines κ at V and the value of κ will be independent of the time t.

Now at any point of the shock line[2]

$$\kappa = -\frac{\partial^2 f/\partial x^2}{[1 + (\partial f/\partial x)^2]^{3/2}}.$$

Hence $\partial^2 f/\partial x^2$ is independent of t at V. This result, when combined with our previous results on the derivatives of the function F, enables us to say that $\partial^2 F/\partial x^i \partial x^j$ is independent of t at V; in fact for i or j equal 3, any such derivative has the value zero. Hence the derivatives $\partial A_\alpha/\partial x^\beta$, $\partial B/\partial x^\alpha$ and $\partial C/\partial x^\alpha$ are independent of t at V; it follows that the equations (2), (3), (4) and (18), (20), (22) determine $\partial u_\alpha/\partial x^\beta$, $\partial p/\partial x^\alpha$ and $\partial \rho/\partial x^\alpha$ independently of t at V. In view of the previous result on the time derivative of the functions we can say that $\partial u_\alpha/\partial x^i$, $\partial p/\partial x^i$ and $\partial \rho/\partial x^i$ are independent of t at V and for $i = 3$ each of these derivatives is equal to zero.

The above procedure can be extended, step for step, to the consideration of the second derivatives of the functions u_α, p and ρ. We will state the case briefly as follows. First

$$\frac{\partial^2 u_\alpha}{\partial x^i \partial t} = \frac{\partial^2 p}{\partial x^i \partial t} = \frac{\partial^2 \rho}{\partial x^i \partial t} = 0$$

from the above result. Hence the equations obtained by differentiating (2), (3) and (4) with respect to x^i are the same, at V, as for steady flow. But $\partial^3 F/\partial x^\alpha \partial x^\beta \partial t = 0$ at V and hence $\partial^2 G/\partial x^\alpha \partial x^\beta = 0$ at V; combining this with the previous result on the derivatives of G we can say that $\partial^2 G/\partial x^i \partial x^j = 0$ at V. Since G, together with its first and second derivatives, relative to the variables x^α, vanishes at V, it follows that the system composed of (25) and the corresponding equations for the functions p and ρ are algebraically the same, at V, as for steady flow. We can therefore deduce the relation (9), for $n = 1$, at V. Hence $\kappa_{(1)}$ is a constant at V, independent of t, since α is non-singular and $G_1(M, \alpha) \neq 0$. But

$$\frac{d\kappa}{ds} = \frac{d\kappa}{dx}\frac{dx}{ds} = \left\{\frac{\partial^3 f/\partial x^3}{[1 + (\partial f/\partial x)^2]^{3/2}} + \cdots\right\} \cos \alpha.$$

Then since, by hypothesis, α does not have the singular value $\alpha = 90°$ [2] it follows from the above equation that $\partial^3 f/\partial x^3$ is independent of t at V. Using this result we see that $\partial^3 F/\partial x^i \partial x^j \partial x^k$ is independent of t at V. Hence all derivatives $\partial^2 A_\alpha/\partial x^\beta \partial x^\gamma, \cdots$ are independent of t at V; it follows that each second derivative of u_α, p and ρ is independent of t at V and whenever differentiation with respect to t is involved the derivative vanishes.

It is evident that this process can be continued to give the following result. The quantities $\kappa, \kappa_{(1)}, \kappa_{(2)}, \cdots$ are independent of t at V when α is non-singular relative to M. Also the functions u_α, p and ρ and their derivatives of any order with respect to the x^i are independent of t at V and any such derivative vanishes when it involves differentiation with respect to t.

9. Completion of proof. Along the shock line $\Sigma(t)$ at any time t we have

$$d\lambda^\alpha/ds = \kappa\nu^\alpha, \qquad \partial\nu^\alpha/ds = -\kappa\lambda^\alpha \tag{28}$$

where λ^α are the components of the unit tangent vector directed as shown in the above figure. Hence

$$dx^\alpha/ds = \lambda^\alpha, \qquad d^2x^\alpha/ds^2 = \kappa\nu^\alpha, \qquad d^3x^\alpha/ds^3 = -\kappa^2\lambda^\alpha + \kappa_{(1)}\nu^\alpha, \cdots. \tag{29}$$

Since the shock line is analytic by hypothesis it is represented in the neighborhood of V by the convergent series

$$x^\alpha = \left(\frac{dx^\alpha}{ds}\right)_V s + \frac{1}{2!}\left(\frac{d^2x^\alpha}{ds^2}\right)_V s^2 + \cdots. \tag{30}$$

But $(dx^\alpha/ds)_V = \lambda^\alpha$ and the components ν^α have fixed values at V, independent of t, since α has such a value (§2); also the values of the following coefficients in the series are determined at V, independently of t, on account of (29) and the result stated at the end of §8. Hence (30) represents a fixed curve in the plane and it follows from this result and the process of analytic continuation that the complete shock line $\Sigma(t)$ does not vary with the time. Similarly it follows from the fact that the quantities u_α, p and ρ have fixed values at V (§7) and the result on the derivatives of these functions at the end of §8 that u_α, p and ρ are independent of t along the shock line and in the region behind this line, i.e. the flow behind the shock line is steady. This completes the proof of the italicized result in §3.

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(Received June 30, 1948)

[2] The minus sign in this equation arises from the orientation of the vectors λ and ν (see above figure) and equations (28) involving κ.

Reprinted from JOURNAL OF MATHEMATICS AND PHYSICS
Vol. XXVIII, No. 2, July, 1949

NUMERICAL INTEGRATION WITH A WEIGHT FUNCTION x

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1. Introduction. The numerical evaluation of definite integrals by approximating sums has been studied by numerous authors. Quite generally one may consider two sets of values $\{c_i\}$ and $\{x_i\}$, $i = 1, 2, 3, \cdots n$, and study the correspondence

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^n c_i f(x_i). \tag{1}$$

Special cases arise when it is desired that the values $\{x_i\}$ be evenly spaced on the interval (a, b), or when the values $\{c_i\}$ are to be equal. The latter case is particularly advantageous when the values $\{f(x_i)\}$ are experimentally determined and subject to error. For if a number of quantities are equally liable to be in error, and if a linear function of these quantities is formed subject to the condition that the sum of the coefficients is fixed, then the probable error of this linear function is least when the coefficients are all equal [1].

A theoretical study of this latter case was made by Chebychef [2], who established the correspondence

$$\int_{-1}^{1} f(x)\,dx \simeq \frac{2}{n}\sum_{i=1}^n f(x_{i,n}) \tag{2}$$

where the values $\{x_{i,n}\}$, $i = 1, 2, \cdots n$ are the zeros of the polynomial part of

$$x^n \exp\left[-\frac{n}{2\cdot 3x^2} - \frac{n}{4\cdot 5x^4} - \frac{n}{6\cdot 7x^6} - \cdots\right] \tag{3}$$

written in descending powers of x. The correspondence (2) is exact whenever $f(x)$ is a polynomial of degree n or less. Thus this case does not afford the accuracy of the well-known method of Gauss, which for n points is exact for $f(x)$ a polynomial of degree $2n - 1$ or less.

Salzer [3] has recently tabulated the zeros of the above-mentioned polynomials for the cases $n = 1, 2, \cdots 7$ and 9. For other values of n the polynomials have roots outside the range $(-1, 1)$, making the use of an integration formula such as (2) impractical if not impossible [4]. Salzer [3] has also given a method based on differences which may be used to estimate the error in (2) when $f(x)$ is not a polynomial of degree n or less.

Chebychef [2] also gave a correspondence

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx \simeq \frac{\pi}{n}\sum_{i=1}^n f(x_{i,n}) \tag{4}$$

where the values $\{x_{i,n}\}$ are the zeros of $\cos(n \arccos x)$. These latter functions are essentially the well-known Chebychef polynomials. If the weight function x be used on the range $(0, 1)$, the corresponding quadrature formula is

$$\int_0^1 xf(x)\,dx \simeq \frac{1}{2n}\sum_{i=1}^n f(x_{i,n}). \tag{5}$$

This formula will be investigated further in sections 2 and 3.

For the range $(-1, 1)$, the possible correspondence relation

$$\int_{-1}^{1} xf(x)\,dx \simeq c_n \sum_{i=1}^{n} f(x_{in})$$

may be investigated. However, if it is desired to have this correspondence exact for polynomials of degree n or less, it must in particular be exact for $f(x) \equiv 1$. The imposition of this condition leads to the requirement that $c_n = 0$, and hence shows that no real utility can be obtained from such a correspondence. Chebychef met this difficulty by considering a more general correspondence

$$\int_{-1}^{1} xf(x)\,dx \simeq k_m \sum_{i=1}^{m} [f(x_{im}) - f(x_{i+m,m})]. \tag{6}$$

Although there are only $(2m + 1)$ values to be selected, it is possible to make (6) exact for polynomials of degree $(2m + 1)$ or less. For, as Chebychef noted, if the Taylor's series expansion for $f(x)$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(2m+1)}(0)x^{2m+1}}{(2m+1)!} + \frac{f^{2m+2}(\xi)x^{2m+2}}{(2m+2)!}, \quad |\xi| < 1$$

be substituted into the left hand side of (6) and integrated out the term in $f(0)$ has zero coefficient. Thus the $(2m + 1)$ values, k_m, $\{x_{im}\}$, $\{x_{i+m,m}\}$ can be used to secure agreement for the terms in $f^{(p)}(0)$, $p = 1, 2, \cdots, 2m + 1$. This is, of course, equivalent to requiring that (6) be exact for polynomials of degree $(2m + 1)$ or less.

The gain in utility of form (6) is offset by the greatly increased difficulty of determining the constant k_m and the values $\{x_{im}\}$ and $\{x_{i+m,m}\}$. Chebychef used a continued fraction method. This will be investigated further in sections 4 and 5.

2. Equations Satisfied by the Roots $\{x_{i,n}\}$ for Range (0, 1). The method used by Chebychef in finding the values $\{x_{i,n}\}$ for the correspondence (2) may be used to determine the polynomials for which the values $\{x_{i,n}\}$ for correspondence (5) are the zeros. Following Chebychef's procedure, one determines the zeros of the polynomial part of

$$\exp\left[2n\int_0^1 r\log(x-r)\,dr\right] = \exp\left[-2n\int_x^{x-1}(x-w)\log w\,dw\right]$$

$$= \exp\left[n\left\{(x^2-1)\log\frac{x}{x-1} - x - \tfrac{1}{2}\right\}\right] \tag{7}$$

$$= x^n \exp\left[-\frac{2n}{1\cdot 3x} - \frac{2n}{2\cdot 4x^2} - \frac{2n}{3\cdot 5x^3} - \frac{2n}{4\cdot 6x^4} - \cdots\right].$$

The polynomial part of (7) is determined by expansion of the exponential in

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for $= \exp[n\{\cdots\}]$ read $= x^n \exp[n\{\cdots\}]$.

powers of $(1/x)$. Note that this is a more complicated form than (3) because the exponential contains all powers of $(1/x)$.

A second method for determining polynomial equations satisfied by the roots is afforded by the use of certain symmetric functions of the roots. In general, if

$$(8) \qquad (x - r_1)(x - r_2) \cdots (x - r_n) = x^n - a_1 x^{n-1} + \cdots + (-1)^n a_n,$$

the coefficients $\{a_i\}$ may be determined in terms of the roots $\{r_i\}$ as follows [5]

$$(9) \qquad a_1 = \sum r_i, \qquad a_2 = \sum_{i<j} r_i r_j, \qquad a_n = r_1 r_2 \cdots r_n.$$

If one introduces the symmetric functions

$$(10) \qquad s_k = \sum_{i=1}^{n} r_i^k, \qquad k = 1, 2, 3, \cdots$$

then the $\{a_i\}$ can be expressed in terms of $\{s_k\}$ by the relation [5]

$$(11) \qquad a_w = \sum_{\Sigma k p_k = w} (-1)^{w+\Sigma p_k} \frac{s_1^{p_1} s_2^{p_2} s_3^{p_3} \cdots}{1^{p_1} \cdot 2^{p_2} \cdot 3^{p_3} \cdots p_1! p_2! p_3! \cdots}.$$

To make specific the use of symmetric functions in the problem under study in this paper, consider for the case n the solution for the values $\{x_{i,n}\}$ (if such exists) of the algebraic system

$$(12) \qquad \int_0^1 x \cdot x^k \, dx = \frac{1}{2n} \sum_{i=1}^{n} x_{i,n}^k, \qquad k = 1, 2, \cdots n.$$

From this formulation it can be shown that integration formula such as (5) are exact whenever $f(x)$ is a polynomial of degree n or less. From (12) one obtains directly

$$(13) \qquad s_k = 2n/(k + 2).$$

Then equation (11) can be used to determine the coefficients $\{a_{w,n}\}$, $w \leq n$, in the polynomials defined by (8), with the respective values of s_k as given by (13).

Thus two methods are available for determining the polynomials for which the values $\{x_{i,n}\}$ are the roots. The first method follows Chebychef's original memoir and consists in taking the polynomial part of (7). The authors desire to express their appreciation to Prof. J. J. Miller who determined the polynomials by this method. The second method is based on symmetric functions of the roots, and was used by the authors to determine the polynomials. As was to be expected, the two sets of polynomials (for $n = 1, 2, \cdots, 10$) so obtained were identical. The tabulation of the first three polynomials appears below:

$$(14) \qquad h_1(x) = x - \tfrac{2}{3}, \qquad h_2(x) = x^2 - \tfrac{4}{3}x + \tfrac{1}{15}, \qquad h_3(x) = x^3 - 2x^2 + \tfrac{6}{7}x - \tfrac{1}{70}.$$

3. Determination of the values $\{x_{i,n}\}$ for Range (0, 1). The real zeros of the polynomials (14) may be determined by refinements of Newton's method. For correspondence (5) to be useful, the zeros must all be real and must line on the interval $0 \leq x \leq 1$. A summary of the results found is given below:

$n = 1$; $x_{1,1} = 0.6666\ 6667$

$n = 2$; $x_{1,2} = 0.4309\ 6441$, $x_{2,2} = 0.9023\ 6893$

$n = 3$; $x_{1,3} = 0.3409\ 7238$, $x_{2,3} = 0.7680\ 6663$, $x_{3,3} = 0.8909\ 6099$

Remark: For $n = 4, 5, 6, \cdots, 10$ the polynomials either had imaginary roots or at least one real outside of the range $(-1, 1)$, or both.

In view of the more or less negative character of the results as regards applicability to numerical integration formula, no cases beyond $n = 10$ were considered. If any great accuracy is desired and the data warrants same, the use of an older type of numerical integration in which the products of the functions x and $f(x)$ at the respective values $\{x_i\}$ are employed would seem preferable to correspondence (5) in which x is treated as a weight function. If the data is none too accurate, then there is no justification for using an "over-accurate" formula, and correspondence (5) for the case $n = 3$ would lighten the work of evaluating the integral considerably.

4. Equations satisfied by the values $\{x_{im}\}$, $\{x_{i+m,m}\}$, k_m for the Range $(-1, 1)$. Defining

$$\begin{aligned} A_m(z) &= (z - x_{1m})(z - x_{2m}) \cdots (z - x_{mm}) \\ B_m(z) &= (z - x_{m+1,m})(z - x_{m+2,m}) \cdots (z - x_{2m,m}), \end{aligned} \tag{15}$$

Chebychef [2] showed that the desired values $\{x_{im}\}$ and $\{x_{m+i,m}\}$ and the coefficient k_m must satisfy a relation of the form

$$\begin{aligned} &\frac{A_m(z)}{B_m(z)} \exp\left(-C_1 z^{-2m-2} - C_2 z^{-2m-4} - \cdots\right) \\ &\quad = \exp\left(\frac{1}{k_m}\int_{-1}^{1} x \log(z - x)\, dx\right) \\ &\quad = \exp\left[-\frac{2}{k_m}\left(\frac{1}{1\cdot 3z} + \frac{1}{3\cdot 5z^3} + \frac{1}{5\cdot 7z^5} + \frac{1}{7\cdot 9z^7} + \cdots\right)\right] \end{aligned} \tag{16}$$

where C_1 and C_2 are constants.

If one expands the right hand side of (16) in the associated continued fraction form [6], the p^{th} convergent will agree with the function thru terms of order $(1/z)^{2p}$. Likewise for the left hand side. But the m^{th} convergent for the left hand side is immediately identifiable as $A_m(z)/B_m(z)$.

Hence one converts the right hand side of (16) to the associated continued fraction, takes the m^{th} convergent and thereby obtains $A_m(z)$ and $B_m(z)$, but as functions of k_m, as yet still unknown. Procedures for carrying this out are given by Perron [7] and Wall [8].

To obtain k_m, the m^{th} complete quotient in the continued fraction may be set equal to zero. In the problem at hand, the associated continued fraction has the form

$$1 + \frac{a_0}{b_1 + z -}\ \frac{a_1}{b_2 + z -}\ \frac{a_2}{b_3 + z -}\ \cdots\ \frac{a_{p-1}}{b_p + z -}\ \cdots. \tag{17}$$

Thus one sets $a_m = 0$. The resulting condition on k_m may then be solved, and when k_m is known $A_m(z)$ and $B_m(z)$ become polynomials with numerical coef-

ficients, and hence their zeros may be determined to give the roots $\{x_{im}\}$ and $\{x_{m+i,m}\}$.

Chebychef [2] gave numerical approximations to five decimals for the case $m = 2$. Radau [9] gave numerical approximations to seven decimals for $n = 2$ and to six decimals for $n = 3$. This paper gives numerical values to eight decimals for $n = 2, 3$ and to seven decimals for $n = 4$. In several cases, Radau's values are found to be in error by more than one unit in his last retained decimal.

The power series expansion of the right hand side of (16) may be written in the form

$$\exp\left[-\frac{2}{k_m}\left\{\frac{1}{1\cdot 3z}+\frac{1}{3\cdot 5z^3}+\cdots\right\}\right]=1+\frac{C_0}{z}+\frac{C_1}{z^2}+\cdots\frac{C_p}{z^{p+1}}+\cdots \tag{18}$$

where the $\{C_i\}$ have the following values:

$$C_0=-\frac{2}{3k},\qquad C_1=\frac{2}{9k^3}\qquad C_2=\frac{-(54k^2+20)}{405k^3},\cdots \tag{19}$$

(The subscript m on k_m has been dropped in the above representation.)

One may use either the method of Perron [7] or that of Wall [8] to obtain the polynomials and complete quotients. Writing the polynomials as $\bar{A}_m(z)$ and $\bar{B}_m(z)$ with integral coefficients in k_m and z rather than as $A_m(z)$ and $B_m(z)$ defined by relation (15), and taking only the numerator $\bar{a}_m$ of the complete quotient, one gets,

$$\bar{a}_1=27k^2-5,\qquad \bar{A}_1(z)=3kz-1,\qquad \bar{B}_1(z)=3kz+1,$$

$$\bar{a}_2=5832k^4-945k^2+35,\qquad \bar{A}_2(z)=135k^2z^2-45kz+5-27k^2,$$

$$\bar{B}_2(z)=135k^2z^2+45kz+5-27k^2.$$

Since $\bar{A}_m(z)$ and $\bar{B}_m(z)$ have the same numerical values for their roots, only one of these polynomials will be written for $m = 3, 4$.

$$\bar{a}_3=2{,}361{,}960k^6-811{,}377k^4+76{,}545k^2-2835k^2+35$$

$$\bar{A}_3(z)=2835k^3(27k^2-5)z^3-945(27k^2-5)z^2+3k(-10{,}935k^4+2835k^2-210)z-(-5832k^4+945k^2-35)$$

$$\bar{a}_4=550{,}998{,}028{,}800k^{12}-175{,}707{,}149{,}184k^{10}+18{,}947{,}643{,}120k^8-977{,}556{,}195k^6+26{,}663{,}175k^4-363{,}825k^2+1925$$

$$\bar{A}_4(z)=(5832k^4-945k^2+35)\,8505k^4z^4-(5832k^4-945k^2+35)\,2835k^3z^3+(-11{,}664k^6+2430k^4-189k^2+5)\,2835k^2z^2-(-367{,}416k^6+54{,}675k^4-2835k^2+50)\,21kz+(2{,}361{,}960k^6-811{,}377k^4+76{,}545k^2-2835k^2+35).$$

5. Values of $\{x_{im}\}$ and k_m for Range $(-1, 1)$. The solutions of equations

$$\bar{a}_m=0,\qquad \bar{A}_m(z)=0,\qquad \bar{B}_m(z)=0 \tag{20}$$

are tabulated below. It was noted by Chebychef that $x_{im} = -x_{m+i,m}$. Further, for each positive root k_m there is also the root $-k_m$. However, correspondence (6) is such that no loss in generality occurs if only positive values of k_m are accepted. For $m = 2, 3, 4$ multiple k values were obtained, and a multiple subscript is used on the k values to distinguish these cases.

$n = 1$; $k_1 = (5/27)^{1/2} = 0.4303\ 3148$, $x_{11} = (3/5)^{1/2} = 0.7745\ 9667 = -x_{21}$.

$n = 2$; $k_{2_1} = 0.2393\ 7152$	$k_{2_2} = 0.3236\ 3299$
$x_{12} = 0.5002\ 9899 = -x_{22}$	$x_{12} = 0.1809\ 2639 = -x_{32}$
$x_{22} = 0.8922\ 3647 = -x_{42}$	$x_{22} = 0.8490\ 4690 = -x_{42}$
$n = 3$; $k_{3_1} = 0.1599\ 1451$	$k_{3_2} = 0.4682\ 8387$
$x_{13} = 0.4429\ 8611 = -x_{43}$	$x_{13} = -0.7636\ 9412 = -x_{43}$
$x_{23} = 0.7121\ 5445 = -x_{53}$	$x_{23} = 0.6125\ 4361 = -x_{53}$
$x_{33} = 0.9293\ 0656 = -x_{63}$	$x_{33} = 0.8629\ 6946 = -x_{63}$
$n = 4$; $k_{4_1} = 0.1223\ 363$	$k_{4_2} = 0.1425\ 864$
$x_{14} = 0.3549\ 416 = -x_{84}$	$x_{14} = 0.1049\ 594 = -x_{84}$
$x_{24} = 0.6433\ 097 = -x_{64}$	$x_{24} = 0.5559\ 632 = -x_{64}$
$x_{34} = 0.7783\ 202 = -x_{74}$	$x_{34} = 0.7378\ 081 = -x_{74}$
$x_{44} = 0.9481\ 574 = -x_{84}$	$x_{44} = 0.9390\ 326 = -x_{84}$
$k_{4_3} = 0.2380\ 212$	$k_{4_4} = 0.4087\ 259$
$x_{14} = -1.1834\ 83 = -x_{54}$	$x_{14} = -0.8288\ 129 = -x_{74}$
$x_{24} = 0.5100\ 129 = -x_{64}$	$x_{24} = 0.0499\ 8867 = -x_{64}$
$x_{34} = 0.8895\ 332 = -x_{74}$	$x_{34} = 0.6998\ 485 = -x_{74}$
$x_{44} = 1.1843\ 73 = -x_{84}$	$x_{44} = 0.8945\ 183 = -x_{84}$

Note that the value k_{4_3} (0.2380 212) requires two x abscissae outside the range of integration $(-1, +1)$ and thus the corresponding formula is of limited utility.

Although not a valid case for $n = 4$, the following case does present some element of interest. In determining the value of a_4, it was found that

$$a_4 = \frac{(5 - 27k^2)^3}{(5 - 27k^2)^2} R(k)$$

where $R(k)$ is a rational function of k.
As $\bar{a}_4^*$, then, one would take

$$\bar{a}_4^* = (5 - 27k^2)\ [\text{numerator of } R(k)],\ = \bar{a}_4(5 - 27k^2) \tag{21}$$

and this would give another k value, k_{4_5} for which

$k_{4_5} = \sqrt{5/27} = 0.4303\ 3148$

$x_{14} = -0.7957\ 3702 = -x_{94}$, $x_{24} = +0.0296\ 2803\ 3 = -x_{84}$,

$x_{34} = +0.6619\ 9633 = -x_{74}$, $x_{44} = +0.8787\ 0933 = -x_{84}$.

However, these values of k_{4_3} and $\{x_{i3}\}$ do not give an accurate result when used in correspondence (6) for polynomials of degree 9 ($=2.4 + 1$), but only for polynomials of degree 8 or less. Thus the division by zero in a_4 for $k_{4_3} = \sqrt{5/27}$ can not be made, and properly speaking k_{4_3} is not an admissible k value.

As checks on the roots $\{x_{im}\}$, the fact that integration formula (6) is exact for polynomials of degree $(2m + 1)$ or less may be used.

For $f(x) = x^j$, there results

$$\frac{1 - (-1)^{j+2}}{j + 2} = \int_{-1}^{1} x \cdot x^j \, dx = k_m \left[\sum_{i=1}^{m} x_{im}^j - \sum_{i=1}^{m} (-x_{im})^j \right].$$

For even values of j, this yields the identities $0 = 0$, for odd j, $j = 1, 3, \cdots 2m + 1$, there results

$$1/(j + 2) = k_m \textstyle\sum_{i=1}^{m} x_{im}^j .$$

Thus there are $(m + 1)$ checks possible on the coefficient k_m and $\{x_{im}\}$. These checks were made on the roots tabulated in this article. As previously noted, the check for $j = 9$ on the k_{4_3} values is not satisfied. Indeed, it was the lack of a check here which prompted a re-examination of the roots of $\bar{a}_4^* = 0$, as given by equation (21).

6. Conclusion. For the range (0, 1), integration correspondence

$$\int_0^1 xf(x) \, dx \simeq \frac{1}{2n} \textstyle\sum_{i=1}^{n} f(x_{i,n}) \tag{5}$$

gives usable relations only for $n = 1, 2, 3$ in the first ten integral cases. On the other hand, for the range $(-1, 1)$, integration correspondence

$$\int_{-1}^{1} xf(x) \, dx \simeq k_m \textstyle\sum_{i=1}^{m} [f(x_{im}) - f(-x_{im})] \tag{6}$$

gives a multiplicity of possible relations for $m = 2, 3, 4$. It would appear that integration correspondence (6) would be of value in determinations of first moments of empirical distribution functions.

As a numerical example, the value of $I = \int_{-1}^{1} x \exp(x) \, dx$ has been computed by four different methods, using six subdivisions of the range $(-1, 1)$. These methods were:

a. *Newton-Cotes method.*

$$I = \int_{-1}^{1} x \exp(x) \, dx \simeq \frac{1}{420} [41g(-1) + 216g(-2/3) + 27g(-1/3) + 272g(0) + 27g(1/3) + 216g(2/3) + 41g(1)] = 0.7357\ 6309$$

where $g(x) = x \exp(x)$.

b. *Equal coefficient but unweighted formula of Chebychef.*

The coefficient 1/3 and $\{x_i\}$ values are given by Salzer [3].

$$\int_{-1}^{1} x \exp(x)\, dx \simeq \tfrac{1}{3} \sum_{i=1}^{6} x_i \exp(x_i) = 0.7352\ 4968.$$

c and d. *Equal Coefficient, Weight Function x Formulae.*
These two cases are given in Section V under the case $n = 3$. There are two values of k and hence two integration cases.

$$I = \int_{-1}^{1} x \exp(x)\, dx = k_3 \sum_{i=1}^{3} [\exp(x_{i3}) - \exp(-x_{i3})].$$

For $k_{3_1} = 0.1559\ 1451$, $I = 0.7357\ 5888$.
For $k_{3_2} = 0.4682\ 8387$, $I = 0.7357\ 5887$.
These four approximate values for I may be compared with the exact value,

$$I = \int_{-1}^{1} x \exp(x)\, dx = \frac{2}{e} \simeq 0.7357\ 5888.$$

The greater accuracy afforded by the formulae with weight function x is apparent in the case of this example.

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(Received September 5, 1948)

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Reprinted from JOURNAL OF MATHEMATICS AND PHYSICS
Vol. XXVIII, No. 2, July, 1949

A METHOD FOR DETERMINING CERTAIN CRITICAL MASSES*

BY HARRY E. GOHEEN

As anyone who has attempted to start a coal fire without kindling wood can testify, a certain critical size of reacting substance is necessary to sustain an exothermic reaction. The magnitude of this size depends on the conductivity of the substance, the temperature of the substance, its heat capacity and density, its rate of heat generation, and its rate of heat loss to the surrounding medium. The last mentioned factor depends on the geometrical shape of the reacting substance, the temperature of the surrounding material, the temperature gradient at the surface and perhaps upon some possibly complicated function which determines the rate of heat loss per unit area from the surface to the surrounding material.

In the following discussion some highly idealized cases will be considered, namely the cases of the sphere, the infinitely long and wide slab and the infinitely long cylinder of hot reacting substance set into a supposedly infinite matrix of the same substance at a temperature too low to sustain the reaction. The discussion is concerned explicitly with reactions such that the rate of heat generation per unit time per unit volume can be approximated by the expression $\lambda e^{-\epsilon/RU}$, λ, ϵ, and R being constants,[1] but the method is such that this can be any analytic function of the temperature, U. The heat capacity, density, and heat conductivity are supposed constant with temperature.

It is to be noted that these idealizations are not too far removed from possible application provided the average internal temperature of the reacting substance can be determined. Since this is certainly a difficult determination, however, great accuracy in the computation of the critical sizes is not required, since the experimental check on the method to great accuracy is not possible. The proof of the convergence of the method proposed is not attempted in the paper as presumably not leading to further insight into the problem.

Under the hypotheses mentioned above, the differential equation governing the temperature distribution in the space consisting of the reacting substance and its cooler matrix is

$$c\rho\partial U/\partial t_1 = k\nabla^2 U + \lambda e^{-(\epsilon/RU)}, \tag{I}$$

with initial conditions at $t_1 = 0$ given by $U = U_0$ at any point inside the hot core and $U = U_1$ at any point outside.

* This paper is an amalgamation and condensation of two papers, "A Non-Rigorous Suggestion for Numerical Solution of Certain Problems in Heat Generation," presented to the American Mathematical Society at the New Haven meeting in September, 1947; and "The Critical Radius of a Heat-Generating Cylinder," presented to the American Mathematical Society at the Christmas meeting in Athens, Georgia, in December, 1947.

[1] See Hottel, H. C., et al., Combustion Rate of Carbon, Journal of Industrial and Engineering Chemistry, 26, 749–757 (1934), page 749; Bamford, C. H., Crank, J. and Malan, D. The combustion of wood, Proceedings of the Cambridge Philosophical Society 42, 166–182 (1946), pages 166–170.

A change of variables,

$$U = \alpha_1 T, \qquad s_1 = \alpha_2 s, \qquad t_1 = \alpha_3 t, \tag{II}$$

t_1 being the measurement of time in which (I) is written and s_1 being the measurement of distance in which (I) is written, yields

$$\partial T/\partial t = \nabla^2 T + e^{-(1/T)}, \tag{III}$$

provided α_1, α_2, and α_3, are given by the equations,

$$\alpha_1 = R/\epsilon, \qquad \alpha_2 = \sqrt{k/\lambda}, \qquad \alpha_3 = c\rho\epsilon/R\lambda. \tag{IV}$$

The initial temperatures U_0 and U_1, are replaced through the change of variable (II) by the initial dimensionless temperatures θ_0 and θ_1 respectively.

For the case in which the hot inner core is a sphere, equation (III) reduces to

$$\partial T/\partial t = \partial^2 T/\partial x^2 + (2/x)\partial T/\partial x + e^{-(1/T(x,t))}, \tag{1.1}$$

x being the dimensionless variable corresponding to the radial distance of a point in the spherically symmetric temperature field. The initial conditions are that at $t = 0$, $T = \theta_0$ if $0 \leqq x < a$ and $T = \theta_1$ if $x > a$.

In an unpublished memorandum C. Loewner has given a method of solution for the above equation and initial conditions, which is of basic importance to the method developed in this paper. Since the paper of Loewner is not published, it will be briefly outlined here. The gist of the method is the fact that it is possible to obtain in closed form the solution when $\theta_0 = \theta_1$ and the solution of the equation for the first perturbation, $\theta_0 - \theta_1$ being taken as parameter. The sum of these two solutions is supposed to satisfy the side conditions imposed on xT and hence the remainder has zero initial values for $0 < x < \infty$, and zero boundary values at $x = 0$, the calculations being thereby materially simplified.

With Professor Loewner's permission, his argument is reproduced below, omitting the essential part; namely that the solution for the remainder exists and under proper hypotheses at infinity is unique.

Equation (1.1) can be simplified by the standard device of considering xT as the dependent variable. If this is done, equation (1.1) becomes

$$\partial(xT)/\partial t = \partial^2(xT)/\partial x^2 + xe^{-(x/(xT))}. \tag{1.2}$$

Suppose that xT in equation (1.2) is a function of θ_0 such that at the value $\theta_0 = \theta_1$, xT can be expanded in powers of $\theta_0 - \theta_1$ by Taylor's Theorem with two terms and a remainder into the form,

$$xT = xT(\theta_1, \theta_1, x, t) + (\theta_0 - \theta_1)\varphi(\theta_1, x, t) + \chi, \tag{1.3}$$

in which the first term is the value of xT when $\theta_0 = \theta_1$, the coefficient of $\theta_0 - \theta_1$ is $(\partial T/\partial\theta_0)_{\theta_0=\theta_1}$, and χ is the remainder.

Now $T(\theta_1, \theta_1, x, t)$ is independent of x, for in this case the temperature throughout the space is θ_1 at time $t = 0$ and since there is no space variation of T at $t = 0$, there will be none for $t > 0$. Then $T(\theta_1, \theta_1, x, t) = w(t)$ in which $w(t)$ is the solution of the ordinary differential equation,

(1.4) $$dw/dt = e^{-(1/w)},$$

with $w(0) = \theta_1$.

To determine $\varphi(\theta_1, x, t)$, differentiate equation (1.2), supposing that the derivatives exist, with respect to θ_0. Since the order of differentiation is immaterial as long as the derivatives are continuous, we obtain

(1.5) $$\partial[x\partial T/\partial\theta_0]/\partial t = \partial^2[x\partial T/\partial\theta_0]/\partial x^2 + 1/T^2 e^{-1/T}(\partial(xT)/\partial\theta_0).$$

Setting $\theta_0 = \theta_1$ in equation (1.5) we obtain

(1.6) $$\partial\varphi/\partial t = \partial^2\varphi/\partial x^2 + (e^{-1/w}\varphi)/w^2,$$

w being the solution to equation (1.4) above.

The substitution, $\varphi = e^{-1/w}\zeta$, transforms equation (1.6) into

(1.7) $$\partial\zeta/\partial t = \partial^2\zeta/\partial x^2.$$

To determine initial values for equation (1.7), we require that the first two terms of equation (1.3) satisfy the initial conditions on T. This means that at $t = 0$, $\zeta = e^{1/\theta_1}x$ if $0 \leqq x < a$ and $\zeta = 0$ if $x > a$. The solution to equation (1.7) subject to these initial values is

(1.8) $$\zeta = \frac{e^{1/\theta_1}}{\sqrt{\pi}} \int_0^a \xi[e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t}]\, d\xi/2\sqrt{t}.$$

Hence

(1.9) $$xT = xw + (\theta_0 - \theta_1)[(e^{-1/w}e^{1/\theta_1})/2\sqrt{\pi t}] \cdot \left\{\int_0^a \xi[e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t}]\, d\xi\right\} + \chi.$$

The remainder, χ, satisfies the differential equation,

(1.10) $$\partial\chi/\partial t = \partial^2\chi/\partial x^2 + xe^{-1/T} - xe^{-1/w} - (\theta_0 - \theta_1)[e^{-1/w}/w^2]\varphi,$$

with initial conditions $\chi(0, t) = \chi(x, 0) = 0$. If we further make the reasonable assumption that at infinity T behaves like w, it is possible to solve equation (1.10) by finite difference technique, taking for infinity two or three times a. Actually much more than this has been done, for as mentioned above, Professor Loewner has proved existence and uniqueness of the solution of (1.10).

As indicated in the introduction, the value of $T(0, t)$ will increase monotonically if a is large enough. For a small, but different from zero, there will be a temporary increase of temperature at $x = 0$, followed by a reduction in temperature to the temperature for $x > a$. The problem is to obtain a critical value, A, such that if $0 < a < A$, the latter case will occur; if $A < a < \infty$, the former will. Since we are interested only in A, it seems unnecessary to compute the entire temperature distribution. The contribution of this paper is a means for avoiding computation except at $x = 0$, thus speeding up by a large factor the obtaining of a value for A. In addition the author develops the perturbation formulae for

the infinite slab and infinite cylinder and applies the methods developed for the sphere, to them.

The temperature at the origin may be obtained as follows. Taking the limit as x approaches 0 in equation (1.1) gives

$$(\partial T/\partial t)_{x=0} = 3(\partial^2 T/\partial x^2)_{x=0} + e^{-(1/T(0,t))}. \tag{1.11}$$

Assume that T and χ are differentiable functions of x with derivatives of all orders with respect to x in the neighborhood of $x = 0$. Then, differentiating equation (1.3) three times with respect to x and setting $x = 0$ give

$$3(\partial^3 T/\partial x^3)_{x=0} = (\theta_0 - \theta_1)(\partial^3\varphi/\partial x^3)_{x=0} + (\partial^3\chi/\partial x^3)_{x=0}. \tag{1.12}$$

In general we have,

$$(2n+1)(\partial^{2n}T/\partial x^{2n})_{x=0} = (\theta_0 - \theta_1)(\partial^{2n+1}\varphi/\partial x^{2n+1})_{x=0} + (\partial^{2n+1}\chi/\partial x^{2n+1})_{x=0}, \tag{1.13}$$

while all odd order derivatives of T with respect to x vanish at $x = 0$. If $\theta_1 << 1$, $w'(t)$ is constant and equal to θ_1. Hence in this case,

$$(\partial^3\varphi/\partial x^3)_{x=0} = -[(\theta_0 - \theta_1)a^3 e^{-a^2/4t}]/4\sqrt{\pi}\, t^{5/2}, \tag{1.14}$$

and since $\varphi = e^{-1/w}\cdot\zeta$ in which ζ satisfies (1.7), we have the following recurrence formula for $(\partial^{2n+1}\varphi/\partial x^{2n+1})_{x=0}$,

$$(\partial^{2n+1}\varphi/\partial x^{2n+1})_{x=0} = d[(\partial^{2n-1}\varphi/\partial x^{2n-1})_{x=0}]/dt, \tag{1.15}$$

with $(\partial^3\varphi/\partial x^3)_{x=0}$ given by equation (1.14).[2]

Since w is considered a small constant, equation (1.10) may be approximated by

$$\partial\chi/\partial t = \partial^2\chi/\partial x^2 + xe^{-1/T}. \tag{1.16}$$

2. With these preliminary considerations, equation (1.11) may be written, using equations (1.12) and (1.14),

$$d[T(0,t)]/dt = e^{-1/T(0,t)} - [(\theta_0 - \theta_1)a^3 e^{-a^2/4t}]/4\sqrt{\pi}\, t^{5/2} + (\partial^2\chi/\partial x^2)_{x=0}. \tag{2.1}$$

Differentiating equation (1.16) three times with respect to x and setting $x = 0$, give

$$d[(\partial^3\chi/\partial x^3)_{x=0}]/dt = (\partial^5\chi/\partial x^5)_{x=0} + 3e^{-1/T}[(\partial^2 T/\partial x^2)_{x=0}]/T^2, \tag{2.2}$$

which, in light of equations (1.12) and (1.14) transforms into

$$d[(\partial^3\chi/\partial x^3)_{x=0}]/dt = (\partial^5\chi/\partial x^5)_{x=0} + [e^{-1/T}/T^2]\{-[(\theta_0 - \theta_1)a^3 e^{-a^2/4t}]/4\sqrt{\pi}\, t^{5/2} + (\partial^3\chi/\partial x^3)_{x=0}\}. \tag{2.3}$$

The process may be continued by successive differentiation of equation (1.16)

[2] If θ_1 is not small enough to make $w(t)$ constant to the accuracy of the computation, each value of $(\partial^{2n+1}\varphi/\partial x^{2n+1})_{x=0}$ obtained from (1.14) by the recurrence formula (1.15) must be multiplied by $e^{-1/w}e^{1/\theta}$. Furthermore (1.16) will have the additional terms—$(\theta_0 - \theta_1)e^{-1/w}\varphi/w^2 - e^{-1/w}$.

with respect to x. Making the assumption that the derivatives exist, we obtain the infinite set of ordinary differential equations for the temperature at the origin and the odd order derivatives of χ with respect to x at $x = 0$,

$$
\begin{aligned}
d[T(0,t)]/dt &= e^{-(1/T)} - [(\theta_0 - \theta_1)a^3 e^{-(a^2/4t)}]/4\sqrt{\pi}\,t^{5/2} + (\partial^3\chi/\partial x^3)_{x=0}\,; \\
d[(\partial^3\chi/\partial x^3)_{x=0}]/dt &= (\partial^5\chi/\partial x^5)_{x=0} \\
&\quad + e^{-(1/T)}/T^2\{-[(\theta_0 - \theta_1)a^3 e^{-(a^2/4t)}]/4\sqrt{\pi}\,t^{5/2} + (\partial^3\chi/\partial x^3)_{x=0}\}\,; \\
d[(\partial^5\chi/\partial x^5)_{x=0}]/dt &= (\partial^7\chi/\partial x^7)_{x=0} \\
&\quad + e^{-1/T}[(1-2T)/T^4](15/9)\{(\partial^3\chi/\partial x^3)_{x=0} - [(\theta_0 - \theta_1)a^3 e^{-a^2/4t}/4\sqrt{\pi}\,t^{5/2}]\}^2 \\
&\quad + [e^{-(1/T)}/T^2]\{e^{-(a^2/4t)}(\theta_0 - \theta_1)[5a^3/2t^{7/2} - a^5/4t^{9/2}]/4\sqrt{\pi}\} \\
&\quad + [e^{-(1/T)}/T^2](\partial^5\chi/\partial x^5)_{x=0}\,; \qquad (2.4)
\end{aligned}
$$

. . . . . .

. . . . . ,

with the initial values at $t = 0$, $T = \theta_0$ and $(\partial^{2n+1}\chi/\partial x^{2n+1})_{x=0} = 0$. The initial conditions on $(\partial^{2n+1}\chi/\partial x^{2n+1})_{x=0}$ arise from the condition that $\chi(x, 0)$ vanishes identically in x.

Under the change of variables $t = a^2 p$; $a^{2n}(\partial^{2n+1}\chi/\partial x^{2n+1})_{x=0} = y_n$ and $T = y_0$ the system (2.4) becomes

$$
\begin{aligned}
dy_0/dp &= a^2 e^{-(1/y_0)} - (\theta_0 - \theta_1)e^{-(1/4p)}/4\sqrt{\pi}\,p^{5/2} + y_1\,; \\
dy_1/dp &= y_2 + a^2 e^{-(1/y_0)}[y_1 - (\theta_0 - \theta_1)e^{-(1/4p)}/4\sqrt{\pi}\,p^{5/2}]; \\
dy_2/dp &= y_3 + a^2 e^{-(1/y_0)}[5/3)(y_1 - (\theta_0 - \theta_1)e^{-(1/4p)}/4\sqrt{\pi}\,p^{5/2})^2(1 - 2y_0)/y_0^4] \\
&\quad + a^2 e^{-(1/y_0)}[y_2 + (\theta_0 - \theta_1)(5/2p^{7/2} - 1/4p^{9/2})e^{-(1/4p)}]; \qquad (2.5)
\end{aligned}
$$

. . . . . .

. . . . . ;

with the initial values at $p = 0$, $y_0 = \theta_0$, and $y_i = 0$, $i > 0$.

For the particular values $\theta_0 = 0.028570$, $\theta_1 = 0.017143$ the set of equations (2.5) setting $y_3 = 0$ and neglecting the rest of the set, has been integrated. The computations are presented in Table I. These particular values of θ_0 and θ_1, are of interest as a test of the method, for they give a critical radius which agrees well with the value obtained at the Bureau of Ordnance Computation Project at Harvard by elaborate difference equation technique.[3] It is to be emphasized

[3] Bloch, R. M., Solution of the Heat Sensitivity Equation in the Case of Spherical Symmetry, Reports Nos. 24 and 26 of the Computation Project of the Bureau of Ordnance, Harvard University, 1946, (unpublished). For descriptions of various possible finite difference approximations and a report on their use to solve a problem similar to the one considered in this paper, see also Crank, J. and Nicholson, P. A. A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. Proceedings of the Cambridge Philo. Society 43, 50–67 (1947).

that mathematical rigor is almost completely absent from the above discussion. The technique of approximating the differential equation by a difference equation is also of doubtful validity but the agreement of the answers given by two such different methods seems to indicate that each is a close approximation to the truth.

The advantage of the method proposed in this paper over the finite difference method is that the latter method requires that a great many mesh points be located in the (x, t) plane and a computation of T be made at each mesh point. The production of such a volume of numbers, including such functions as $e^{-1/T}$, is an almost hopeless task for a hand computation, whereas the computations indicated here are capable of giving a result in a few days using only a hand machine and a book of logarithms.

TABLE I

*Spherical Symmetry**

$\theta_0 = 0.028570$; $\theta_1 = 0.017143$ $\log_{10} a = 6.70$

ρ	y_0	y_1	y_2
.00	.0286	.0000	.0000
.01	.0287	.0000	.0000
.02	.0289	.0000	.0000
.03	.0292	.0000	−.0156
.04	.0295	.0011	−.2021
.05	.0298	−.0089	−.6758
.06	.0302	−.0372	−.8357
.07	.03054	−.1197	—
.08	.03051	—	—

* Increasing $\log_{10}a$ by 0.05 resulted in monotonic increase of y_0. Hence for these values of θ_0 and θ_1 the value of $\log_{10}A$ is between 6.70 and 6.75, a result obtained also by the solution of the problem by finite differences.

3. The partial differential equation, (III), for the case of the infinite slab is

$$\partial T/\partial t = \partial^2 T/\partial x^2 + e^{-(1/T)}, \tag{3.1}$$

with boundary conditions at $t = 0$, $T = \theta_0$ if $|x| < a$, $T = \theta_1$ if $|x| > a$.

By use of an argument similar to the argument of Loewner referred to in Section 1 above the solution of equation (3.1) may be reduced to

$$T = w + [(\theta_0 - \theta_1)e^{-(1/w - 1/\theta_1)}]/2\sqrt{\pi t} \int_{-a}^{a} e^{-[(x-\xi)^2/4t]}\, d\xi + \chi \tag{3.2}$$

in which χ satisfies

$$\partial\chi/\partial t = \partial^2\chi/\partial x^2 + e^{-(1/T)} - e^{-1/w} - (\theta_0 - \theta_1)e^{-(2/w - 1/\theta_1)}/w^2\sqrt{4\pi t} \int_{-a}^{a} e^{-[(x-\xi)^2/4t}\, d\xi \tag{3.3}$$

with boundary conditions $\chi(x, 0) = \chi(\pm\infty, t) = 0$.

Neglecting the terms involving $e^{-1/w}$, equation (3.3) can be written in a form similar to that of equation (1.16),

$$\partial\chi/\partial t = \partial^2\chi/\partial x^2 + e^{-(1/T)}. \tag{3.4}$$

By the same method as that of Section 2, the determination of a critical half-width is reduced to a consideration of the situation on the center-line of the slab. As in Section 2, there is obtained an infinite set of ordinary differential equations,

$$\begin{aligned} d[T(0,t)]/dt &= e^{-(1/T)} - (\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}ae^{-(a^2/4t)}/2\sqrt{\pi}t^{3/2} \\ &\quad + (\partial^2\chi/\partial x^2)_{x=0}; \\ d[(\partial^2\chi/\partial x^2)_{x=0}]dt &= (\partial^4\chi/\partial x^4)_{x=0} \\ &\quad + e^{-(1/T)}/T^2[-(\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}ae^{-(a^2/4t)}/2\sqrt{\pi}t^{3/2} + (\partial^2\chi/\partial x^2)_{x=0}]; \\ &\cdots\cdots \\ &\cdots\cdots. \end{aligned} \tag{3.5}$$

By the change of variable $a^{2n}(\partial^{2n}\chi/\partial x^{2n})_{x=0} = y_n$, $(n > 0)$; $T(0, t) = y_0$; and $t = a^2p$, the set may be put into a form similar to that of the system (2.5),

$$\begin{aligned} dy_0/dp &= a^2e^{-(1/y_0)} - (\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}e^{-(1/4p)}/2\sqrt{\pi}p^{3/2} + y_1; \\ dy_1/dp &= a^2e^{-(1/y_0)}/y_0^2[y_1 - (\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}e^{-(1/4p)}/2\sqrt{\pi}p^{3/2}] + y_2; \\ &\cdots\cdots \\ &\cdots\cdots; \end{aligned} \tag{3.6}$$

with $y_0 = \theta_0$ and $y_i = 0$ $(i = 1, 2, 3, \cdots)$ at $p = 0$.

4. For the case in which the hot core is an infinite cylinder of radius a, the fundamental differential equation III becomes

$$\partial T/\partial t = \partial^2 T/\partial r^2 + (1/r)\partial T/\partial r + e^{-(1/T)}, \tag{4.1}$$

with initial values at $t = 0$, $T = \theta_0$ if $r < a$, $T = \theta_1$ if $r > a$. By use of the perturbation technique the solution to (4.1) is seen to be

$$T = \varphi_0(r, t) + (\theta_0 - \theta_1)\varphi + \chi \tag{4.2}$$

with φ_0 constant in r and as a function of t, the solution of the ordinary differential equation

$$dw/dt = e^{-(1/w)}, \tag{4.3}$$

with initial condition $w(0) = \theta_1$. The function $\varphi(r, t)$ is the solution of the partial differential equation

$$\partial\varphi/\partial t = \partial^2\varphi/\partial r^2 + (1/r)\partial\varphi/\partial r + e^{-(1/w)}\varphi/w^2. \tag{4.4}$$

Upon setting $\varphi = e^{-1/w}\zeta$, we have, as in the simpler case of spherical symmetry, the equation

$$\partial\zeta/\partial t = \partial^2\zeta/\partial r^2 + (1/r)\partial\zeta/\partial r. \tag{4.5}$$

For initial values for equation (4.5) we take such values as to make initial values zero for χ, i.e., such values that the first two terms of the right hand side of equation (4.2) satisfy the initial conditions on T. This means that inside the cylinder at $t = 0$, $\zeta = {}^{1/\theta_1}$ and outside it at $t = 0$, $\zeta = 0$.

The solution to equation (4.5) under these conditions is given by the integral,[4]

$$(4.6)\quad \zeta(r, t) = (1/\pi^{3/2}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(x-\lambda)^2+(y-\mu)^2+(z-\nu)^2]/4t} f(\lambda, \mu, \nu)\, d\lambda\, d\mu\, d\nu/8t^{3/2};$$

in which $f(\lambda, \mu, \nu)$ is e^{1/θ_1} if (λ, μ, ν) is a point of the cylinder and is zero otherwise. Transforming to cylindrical coordinates and substituting the value of $f(\lambda, \mu, \nu)$ into equation (4.6) yield

$$(4.7)\quad \zeta(r, t) = e^{1/\theta_1}/\pi^{3/2} \int_{-\infty}^{\infty} e^{-(z-\nu)^2/4t}/2\sqrt{t}\, d\nu \int_0^a \int_0^{2\pi} e^{-(r^2-2r\rho\cos\theta+\rho^2)/4t} \rho\, d\rho\, d\theta/4t,$$

and

$$(4.8)\quad \varphi(r, t) = e^{-(1/w-1/\theta_1)}/4\pi t \int_0^a \int_0^{2\pi} e^{-(\delta^2/4t)} \rho\, d\rho\, d\theta,$$

in which $\delta^2 = r^2 - 2r\rho\cos\theta + \rho^2$.

It is unnecessary to integrate equation (4.8), for we are interested only in $(\partial^2\varphi/\partial r^2)_{r=0}$. In equation (4.1) let r approach 0. Then we have

$$(4.9)\quad d[T(0, t)]/dt = 2(\partial^2 T/\partial r^2)_{r=0} + e^{-(1/T)},$$

which becomes after using equation (4.2) and the fact that $\varphi_0(r, t)$ is independent of r,

$$(4.10)\quad d[T(0, t)]/dt = (\theta_0 - \theta_1)(\partial^2\varphi/\partial r^2)_{r=0} + (\partial^2\chi/\partial r^2)_{r=0} + e^{-(1/T)}.$$

To evaluate $(\partial^2\varphi/\partial r^2)_{r=0}$ we note that it is given by setting $r = 0$ in

$$(4.11)\quad \partial^2\varphi/\partial r^2 = e^{-(1/w-1/\theta_1)}/4\pi t \int_0^a \int_0^{2\pi} e^{-(\delta^2/4t)} \cdot \left[\left(\delta/2t\,\frac{\partial\delta}{\partial r}\right)^2 - \partial\left(\delta/2t\,\frac{\partial\delta}{\partial r}\right)\Big/\partial r\right] \rho\, d\rho\, d\theta,$$

or since $\delta\partial\delta/\partial r = r - \rho\cos\theta$ and $\partial(\delta\partial\delta/\partial r)/\partial r = 1$ in

$$(4.12)\quad \partial^2\varphi/\partial r^2 = e^{-(1/w-1/\theta_1)}/4\pi t \int_0^a \int_0^{2\pi} e^{-(\delta^2/4t)}[(r - \rho\cos\theta)^2/4t^2 - \tfrac{1}{2}t]\rho\, d\rho\, d\theta,$$

to give

$$(4.13)\quad (\partial^2\varphi/\partial r^2)_{r=0} = e^{-(1/w-1/\theta_1)}/4\pi t \int_0^a \int_0^{2\pi} e^{-(\rho^2/4t)}[\rho^2\cos^2\theta/4t^2 - \tfrac{1}{2}t]\rho\, d\rho\, d\theta,$$

[4] See Goursat, E., Cours d'analyse mathematique, vol. 3, 3rd ed. Paris (1923) pp. 107–109.

TABLE II

*Cylindrical Symmetry**

1. $\theta_0 = .05$; $\theta_1 = .01$; $\log_{10}a = 3.59$

β	y_0	y_1	y_2
.00	.0000	.0000	.0500
.01	.0000	.0000	.0503
.02	−.0068	−.0001	.0507
.03	−.0967	−.0014	.0511
.04	−.1175	−.0051	.0514
.05	−.1868	−.0134	.0515
.06	.6492	−.0188	.0514
.07	2.1209	.0057	.0514

2. $\theta_0 = .05$; $\theta_1 = .02$; $\log_{10}a = 3.53$

β	y_0	y_1	y_2
.00	.0000	.0000	.0500
.01	.0000	.0000	.0502
.02	−.0037	.0000	.0505
.03	−.0515	−.0008	.0508
.04	−.1613	−.0036	.0510
.05	−.3356	−.0102	.0510
.06	−.2022	−.0179	.0508

3. $\theta_0 = .04$; $\theta_1 = .02$; $\log_{10}a = 4.53$

β	y_0	y_1	y_2
.00	.0000	.0000	.0400
.01	.0000	.0000	.0402
.02	−.0027	.0000	.0404
.03	−.0384	−.0006	.0406
.04	−.1199	−.0027	.0407
.05	−.1716	−.0068	.0407
.06	−.0486	−.0115	.0406

4. $\theta_0 = .04$; $\theta_1 = .02$; $\log_{10}a = 4.62$

β	y_0	y_1	y_2
.00	.0000	.0000	.0400
.01	.0000	.0000	.0402
.02	−.0061	−.0001	.0405
.03	−.0892	−.0014	.0408
.04	−.3121	−.0067	.0410
.05	−.3119	−.0169	.0411
.06	.7336	−.0232	.0410

* In each of the cases, indicated, increasing $\log_{10}a$ by 0.10 resulted in monotonic increase in y_0. Hence for the cylinder, for the values of θ_0 and θ_1 indicated, $\log_{10}A$ is between the value of $\log_{10}a$ indicated and that value increased by 0.10. For further details of the computation of these tables see the University of Delaware Master's Thesis of Mr. Gilbert Kaskey, June, 1948.

which integrates into

$$
\begin{aligned}
(\partial^2\varphi/\partial r^2)_{r=0} &= e^{-(1/w-1/\theta_1)}/4t \int_0^a e^{-(\rho^2/4t)}[\rho^2/4t^2 - 1/t]\rho\, d\rho \\
&= e^{-(1/w-1/\theta_1)}/t \int_0^{a/2\sqrt{t}} e^{-v^2}(v^3 - v)\, dv \\
&= -e^{-(1/w-1/\theta_1)}a^2e^{-(a^2/4t)}/8t^2.
\end{aligned} \tag{4.14}
$$

To evaluate $(\partial^2\chi/\partial r^2)_{r=0}$, we have the differential equation for χ obtained by substitution from equation (4.2) into equation (4.1),

$$
\begin{aligned}
\partial\chi/\partial t = \partial^2\chi/\partial r^2 + (1/r)\partial\chi/\partial r + e^{-(1/T)} - e^{-(1/w)} \\
- (\theta_0 - \theta_1)e^{-(1/w)}\varphi/w^2,
\end{aligned} \tag{4.15}
$$

with initial condition $\chi(r, 0) \equiv 0$. In virtue of the initial condition on χ, we note that $(\partial^2\chi/\partial r^2)_{r=0}$ is equal to 0 at time $t = 0$. To get its value for $t > 0$, differentiate equation (4.15) twice with respect to r and then let r approach 0. There results

$$
\begin{aligned}
d[2(\partial^2\chi/\partial r^2)_{r=0}]/dt = (8/3)(\partial^4\chi/\partial r^4)_{r=0} + e^{-(1/T)}/T^2[2(\theta_0 - \theta_1)(\partial^2\varphi/\partial r^2)_{r=0} \\
+ 2(\partial^2\chi/\partial r^2)_{r=0}] - 2(\theta_0 - \theta_1)e^{-1/w}/w^2(\partial^2\varphi/\partial r^2)_{r=0}.
\end{aligned} \tag{4.16}
$$

By multiplying equation (4.15) by $\frac{8}{3}$, differentiating four times with respect to r and letting r approach 0, an ordinary differential equation for $(8/3)(\partial^4\chi/\partial r^4)_{r=0}$ may be obtained, which will involve the sixth derivative of χ with respect to r at $r = 0$. In this fashion an infinite set of ordinary differential equations may be constructed for T and $k_n(\partial^{2n}\chi/\partial r^{2n})_{r=0}$[5] the first two of which are, written in the variables $T = y_0$, $k_n(\partial^{2n}\chi/\partial r^{2n})_{r=0} = y_n (n = 1, 2, \cdots)$, $t = a^2p$,

$$
\begin{aligned}
dy_0/dp &= a^2e^{-(1/y_0)} - (\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}e^{-(1/4p)}/4p^2 + y_1 \\
dy_1/dp &= y_2 + a^2e^{-(1/y_0)}/y_0^2[y_1 - (\theta_0 - \theta_1)e^{-(1/w-1/\theta_1)}e^{-1/4p}/4p^2] \\
&\quad - a^2(\theta_0 - \theta_1)e^{-(2/w-1/\theta_1)}/w^2e^{-(1/4p)}/4p^2.
\end{aligned} \tag{4.17}
$$

The system (4.17) truncated at $n = 2$ and with y_2 set equal to zero, has been integrated numerically and the critical radii corresponding to several different values of θ_0 and θ_1 have been determined. Results of this computation appear in Table II.

Acknowledgements. The author wishes to thank the Office of Naval Research for support during part of the investigation, Dr. Raymond J. Seeger for suggesting the problem in the case of spherical symmetry, Professor Charles Loewner for allowing the use of his unpublished work, Messrs. Martin Schwartz and Gilbert Kaskey for carrying out the computations, and Dr. Brooks J. Lockhart for assistance with some of the manipulative complications.

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(Received August 18, 1948)

[5] It can be proved that $k_n = \frac{2}{1}\cdot\frac{4}{2}\cdot\frac{6}{3}\cdots\frac{2n}{2n-1}\cdot a^{2n}$.

Reprinted from JOURNAL OF MATHEMATICS AND PHYSICS
Vol. XXVIII, No. 2, July, 1940

ON THE SUBSONIC FLOW THROUGH CIRCULAR AND STRAIGHT LATTICES OF AIRFOILS

BY C. C. LIN

1. Introduction. In the study of gas turbines and axial flow compressors, one often approximates the situation by considering a two-dimensional flow past an infinite lattice of identical airfoils (Fig. 1). In the centrifugal compressor, a vaned-diffusor is usually installed around the impellor to remove the circulation of the flow. There arises then the problem of flow through a group of identical airfoils arranged regularly around a circle (Fig. 2). The problem of incompressible flow is well-known in the first case, but does not seem to be extensively treated in the second case. In practice, the compressibility of the air can hardly be neglected, and the problem becomes very complicated in either case. The purpose of this paper is to try to provide methods for calculating flows of an incompressible and a compressible fluid through circular and straight lattices.

The difficulty in these cases can be appreciated by considering the much simpler problem of compressible flow past a single airfoil. The only exact analytical method known is the hodograph method, which is however rather lengthy. To bring out the essential aerodynamic features, one usually introduces suitable approximations. The most convenient of them are the Prandtl-Glauert approximation for nearly parallel flows and the von Kármán-Tsien approximation. The Janzen-Rayleigh approximation is suitable only for small Mach numbers. In the present cases, the Prandtl-Glauert approximation is also not suitable, because the flow involved is usually not nearly parallel. Even in the case of the straight lattice, there is often a finite deflection of the direction of flow.

It is thus obvious that the only feasible method is the hodograph method with the linear pressure-volume relation. The validity of this method does not depend on the nearly parallel nature of the flow, but on the fact that the compression involved is never too great. This method has been known for some time, but the complete formulation for flows with circulation has been solved only recently by a slight generalization of the original ideas.[1,2] It turns out that this generalization throws a new light on the problem, and the method can be now easily applied to flows through multiply connected regions, such as those we are concerned with here.

Realizing the complication involved, we are here to present the general inverse method for generating such flows. The direct method requires very complicated calculations, even in the case of an incompressible flow past a single thick airfoil. The present method is general in the sense that the final results involve an arbitrary function which may be used at our disposal to take care of the shape of the airfoils. It thus has great flexibility to be adopted to any design problem. On the other hand, no specific effort has been made in this paper to formulate a design procedure, although it is not expected to be difficult to do so for any set of design conditions.

It might be pertinent to add one word to the line of argument which led to the

FIG. 1. THE STRAIGHT LATTICE

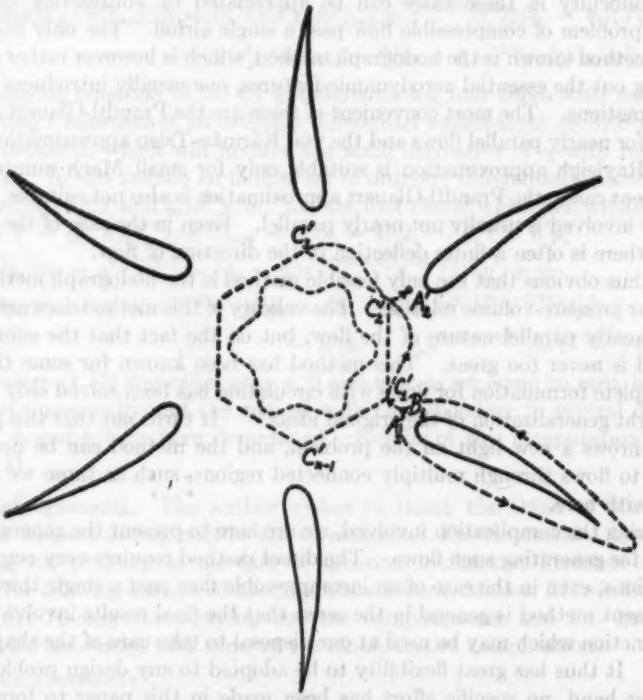

FIG. 2. THE CIRCULAR LATTICE

method. It is emphasized that the mapping problem and the flow problem are entirely separated; e.g., the conformal mapping of a circle into a straight lattice

of identical airfoils is a purely geometrical problem. By examining the most general of such mapping functions, one can easily see the essential part of its nature. The generalization to the circular lattice of a finite number of airfoils is then very easy. By combining these ideas with the recent improvement of the Kármán-Tsien approximation, the method presented below is evolved.

It is almost needless to say that the method is limited to the subsonic range, —an inherent restriction in the linear pressure-volume approximation (when the flow has a stagnation point).

In the next section (§2), we shall first discuss the general characteristics of flows through lattices of airfoils, both circular and straight. It will be followed by a summary of our methods for their calculation (§3). The proof of these methods and the discussion of their consequences will be respectively treated in §4 and §5. In §6, alternative methods will be discussed.

2. General nature of flows through lattices. Before giving the details of our method, let us first say a few words about the general nature of the flows through lattices.

The incompressible flow past a straight lattice of an infinite number of airfoils is well-known. The flow is uniform at infinity to the left, and becomes again uniform at infinity to the right, with however a generally different magnitude and direction. This change in magnitude and direction is related to the circulation around the airfoils. While the circulation around a single airfoil cannot produce any change of the magnitude and direction of flow at infinity, the aggregate influence of an infinite number of them does. The amount of circulation is determined by the Kutta-Joukowski condition at the trailing edge. The subsonic compressible flow does not differ essentially from the incompressible case in its general aspects. The restriction to finite velocity at the trailing edge is however a necessity in this case, instead of an added condition.

The general character of the incompressible flow past a circular lattice may be described as follows. At the center of symmetry, the flow essentially has the nature of a complex source. The circulation may or may not be zero. A contour enclosing all the airfoils at large distances encloses the same source strength, but would in general have a different circulation, the difference being equal to the sum of the circulation around the individual airfoils. The flow at large distances is therefore again essentially that due to a vortex and a source. The circulation around individual airfoils is determined by the Kutta-Joukowski condition, as in the case of straight lattices.

In the compressible case, the flow near the center of symmetry must be changed, because there cannot be a point source or vortex in compressible flow. However, this does not cause any serious difficulty in the application to diffusors, because the central portion is occupied by the impeller itself. We start out with a certain flow with circulation and a radial velocity, at a finite distance from the center. The outside part of the flow does not differ essentially from that in the incompressible case. The circulation around the individual airfoils is again determined by the Kutta-Joukowski condition, and the flow at large distances is

essentially that due to a vortex and a source. In diffusors, it is often required that the circulation outside be zero.

On the other hand, the continuation of the outside part of the flow inwards to the center of symmetry leads to the limiting line at some distance from it. The solution then ceases to exist for the inside part, and further continuation of the solution turns back to the outside part, and leads to the second branch of a multiple-valued solution.

3. Methods for calculating flows through lattices of airfoils. In the present section, we shall present the methods for calculating the flows discussed in the last section. We shall formulate below the method in full for the case of compressible flow through a circular cascade. The other three cases can all be obtained by simplifications of this case, as will be explained below.

Let p, ρ, u, v be the pressure, density and components of velocity of a steady two-dimensional irrotational flow in the x, y plane. Let p be a function of the density ρ only (given either by the isentropic relation or any other approximate relation). Then there exist the velocity potential ϕ and the stream function ψ defined by the following differential relations:

$$d\phi = udx + vdy, \tag{3.1}$$

$$d\psi = -\rho vdx + \rho udy. \tag{3.2}$$

The velocity components u, v and the density ρ are further connected by Bernoulli's equation

$$\tfrac{1}{2}q^2 + \int dp/\rho = \text{const.}, \quad (q^2 = u^2 + v^2). \tag{3.3}$$

It is convenient to refer the density of the gas to that at the stagnation point and to refer all the velocities to the velocity of sound at stagnation. The coordinates x, y may be regarded as referred to the size of the body, and the pressure as referred to the product of stagnation density and the square of the velocity of sound at stagnation. Throughout this article, this process shall be implied, and all the quantities under discussion are dimensionless.

As is well-known, the problem simplifies greatly if the pressure-density relation is approximated by

$$p = A - B/\rho. \tag{3.4}$$

This is the basis of the method of Chaplygin, von Kármán and Tsien. A discussion of its physical interpretation may be found in the papers of these authors. Equation (3.3) leads to

$$c = 1/\rho = \sqrt{1 + q^2}, \tag{3.5}$$

where c is the local velocity of sound. Indeed, B must be equal to unity if the reference stagnation quantities are calculated from (3.3) by the use of (3.4).

Under the approximation (3.4), the following method may be used for constructing two-dimensional subsonic flows through lattices.

Let $F(\zeta)$ be the complex potential function of the incompressible flow due to two complex sources in the ζ-plane ($\zeta = \xi + i\eta$) at $\zeta = a_1$ and $\zeta = a_2$ outside the circular boundary $|\zeta| = 1$ (Fig. 3):

$$F(\zeta) = A_1 \log(\zeta - a_1) + \bar{A}_1 \log(\zeta - 1/\bar{a}_1) + B_1 \log(\zeta - a_2) + \bar{B}_1 \log(\zeta - 1/\bar{a}_2), \quad (3.5)$$

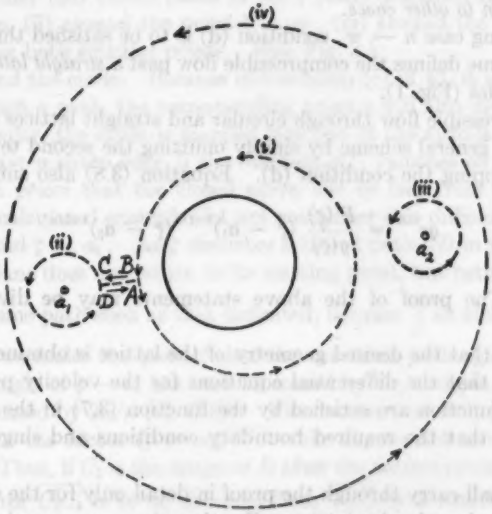

FIG. 3. THE ζ-PLANE, WITH THE FOUR IMPORTANT CONTOURS OF INTEGRATION

where A_1 and B_1 are complex constants, and $\mathrm{Re}\, A_1 = -\mathrm{Re}\, B_1$. The mapping between the z-plane ($z = x + iy$) and the ζ-plane defined by

$$dz = g(\zeta)(\zeta - a_1)^{-1+1/n}(\zeta - a_2)^{-1-1/n}d\zeta - \overline{\tfrac{1}{4}F'^2(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n}(\zeta - a_2)^{1+1/n}d\zeta} \quad (3.6)$$

gives a compressible flow, with the linear pressure volume approximation, past a circular lattice of n identical blades in the z-plane (Fig. 2) with velocity potential ϕ and stream function ψ given by

$$\phi + i\psi = F(\zeta), \quad (3.7)$$

provided that $g(\zeta)$ is chosen to satisfy the following general requirements:

(a) It is regular in the region R, defined by $|\zeta| \geqq 1$.

(b) It does not vanish in R except possibly at one point on the circle, where $F'(\zeta) = 0$; the order of the zero does not exceed one.

(c) Along the circle $|\zeta| = 1$, $\oint dz = 0$.

(d) Except in the neighborhood of $\zeta = a_1$, the following inequality holds in R:

$$|F'(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n}(\zeta - a_2)^{1+1/n}| < 2.$$

Condition (d) sets a limit to the region of validity of the calculated flow field in the z-plane. The magnitude q and direction θ of the velocity at any point is given by

$$\frac{2q}{1+\sqrt{1+q^2}}\,e^{-i\theta} = \frac{F'(\zeta)}{g(\zeta)}\,(\zeta-a_1)^{1-1/n}(\zeta-a_2)^{1+1/n} \tag{3.8}$$

Simplification to other cases.

In the limiting case $n \to \infty$, condition (d) is to be satisfied throughout R, and the above scheme defines the compressible flow past a *straight lattice* of an infinite number of blades (Fig. 1).

The incompressible flow through circular and straight lattices can be obtained from the same general scheme by simply omitting the second term in dz (in Eq. (3.6)) and dropping the condition (d). Equation (3.8) also simplifies to

$$qe^{-i\theta} = \frac{F'(\zeta)}{g(\zeta)}\,(\zeta-a_1)^{1-1/n}(\zeta-a_2)^{1+1/n} \tag{3.9}$$

4. Proof. The proof of the above statements may be divided into three parts:

(α) to show that the desired geometry of the lattice is obtained,

(β) to show that the differential equations for the velocity potential and the stream function are satisfied by the function (3.7) in the z-plane,

(γ) to show that the required boundary conditions and singularities are fulfilled.

Again, we shall carry through the proof in detail only for the case of compressible flow through a circular lattice. For the other cases, we shall point out the modifications necessary.

(α) *Geometry of the mapping.*

To examine the nature of the mapping (3.6), let us first examine its Jacobian for the singularities. The Jacobian

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \operatorname{Im}\left\{\frac{\partial\bar{z}}{\partial\xi}\cdot\frac{\partial z}{\partial\eta}\right\}$$

can be easily shown to be

$$J = \{\,|\,g\,|\cdot|\,\zeta-a_1\,|^{-1+1/n}\,|\,\zeta-a_2\,|^{-1-1/n}\}^2 \cdot\left\{1-\frac{1}{16}\left|\frac{F'(\zeta)}{g(\zeta)}\,(\zeta-a_1)^{1-1/n}(\zeta-a_2)^{1+1/n}\right|^4\right\}. \tag{4.1}$$

Thus, if the condition (d) of §3 is satisfied, the Jacobian will always remain *non-negative* in R. It may vanish at a single point on the circle, which point is then mapped into a corner or a cusp. This is the point where $g(\zeta) = 0$. It is necessary that the condition (b) of §3 be satisfied in order that condition (d) may be satisfied in the neighborhood of this point. In specifying condition (d), it is necessary that a neighborhood of $\zeta = a_1$ be excluded, because the function $F'(\zeta)\{g(\zeta)\}^{-1}(\zeta-a_1)^{1-1/n}(\zeta-a_2)^{1+1/n}$ becomes infinite as $\zeta \to a_1$.

The Jacobian becomes infinite as $\zeta \to a_1$ or $\zeta \to a_2$. As the point $\zeta = a_1$ is to be excluded, we need only pay our attention to the infinity as $\zeta \to a_2$. However, notice that the mapping (3.6) does exhibit branch-point nature both around $\zeta = a_1$ and $\zeta = a_2$.

With these behaviour of the mapping function in mind, let us trace out the actual mapping of the circle $|\zeta| = 1$ into the circular lattice in the z-plane. To do this, we consider four closed paths in the ζ-plane described respectively: (i) around the circle, (ii) around the point $\zeta = a_1$, (iii) around the point $\zeta = a_2$, and (iv) enclosing both singular points. (See Fig. 3.)

(i) Path around the circle. Because of condition (c) of §3, it is clear that, as ζ runs around such a path, the corresponding point z will also describe a closed path. In particular, the circle is mapped into one of the airfoils desired. The boundary will have a sharp edge, if the function $g(\zeta)$ vanishes on the circle. It is not obvious *a priori* that the closed curve will be free from double points. However, some calculated example did not encounter this difficulty.

(ii) Path around $\zeta = a_1$. As ζ describes a closed path CD in the ζ-plane, its image in the z-plane does not return to its starting point, but rather moves from C_1' to C_2' along some path such as that indicated, because $\int dz$ along such a path is not zero. If we make the point ζ describe the same path again, it must be noted that the angle of $\zeta - a_1$ has been increased by 2π, and hence the angle of dz has been increased by $2\pi/n$ (cf. (3.6)). The first term of (3.6) gets this factor $e^{2\pi i/n}$ by virtue of the factor $(\zeta - a_1)^{-1+1/n}$ while the second term gets it from $(\zeta - a_1)^{-1-1/n}$. Thus, if C_3' is the image of D after the second circling of the point $\zeta = a_1$, the vector $\overline{C_2'C_3'}$ is to be obtained from $\overline{C_1'C_2'}$ by a rotation through an angle of $2\pi/n$. In fact, the whole path $C_2'C_3'$ may be obtained from $C_1'C_2'$ this way. Thus, after the path CD has been described n-times, we have a series of points C_1', C_2', $\cdots$ C_{n-1}', C_n', which are the vertices of a regular polygon of n sides, if C_n' coincides with C_1'. To show that this is the case, we introduce an auxiliary variable

$$\zeta_1 = (\zeta - a_1)^{1/n}$$

and consider the neighborhood of the point $\zeta_1 = 0$. Describing the path CD n times is equivalent to a closed path around the origin of the ζ_1-plane. It can be easily seen that the first term in (3.6) becomes regular in ζ_1, and that the second term has a pole with a Laurent development of the form $\zeta_1^{-2}(\alpha + \beta\zeta_1^n + \cdots)$. The residue around $\zeta_1 = 0$ is therefore zero if $n > 1$. Hence, a closed path around $\zeta_1 = 0$ leads to a closed path in the z-plane.

Thus, the mapping of a path CD is obtained by first tracing the curve $C_1'C_2'$ and then rotating the whole figure through an angle of $2\pi/n$ about a certain point n times. Although this center of rotation will not be reached by our mapping, it is fixed and the same for *all* points. To see this, it is only necessary to note that the path CD is an arbitrary path and therefore may be made to cover any point.

The net effect of describing a path around $\zeta = a_1$ is therefore to rotate the

whole picture through an angle of $2\pi/n$ in the same sense as the path is taken in the ζ-plane. Thus n identical airfoils are generated by the circle $|\zeta| = 1$ if ζ keeps on describing closed paths like $ABCDA$ (in the sense indicated in Fig. 3) with the points A, B on the circle.

(iii) *Path around* $\zeta = a_2$. It is clear that when ζ describes a closed path around $\zeta = a_2$, the effect is similar. The only difference is that the rotation through an angle $2\pi/n$ in the z-plane is in a sense opposite to that of the motion of the point ζ. The question arises whether the rotation takes place around the same center. This is equivalent to say whether the effect of describing a path around $\zeta = a_2$ exactly cancels that of describing a path around $\zeta = a_1$ in the same sense. As path (i) gives a nil effect, the question is really whether path (iv) also gives a nil effect. This is indeed the case, as shown below.

(iv) *Path enclosing the circle and the singularities.* To examine this path, we concentrate our attention to the behaviour of (3.6) at the point of infinity. In the first place, the coefficients are single-valued in this neighborhood, as can be easily seen. Let us examinate whether there is a finite residue.

Since $g(\zeta)$ does not vanish as $\zeta \to \infty$,

$$g(\zeta) = g(\infty) + 0(\zeta^{-1}). \tag{4.2}$$

The behaviour of $F'(\zeta)$ at infinity is given by

$$F'(\zeta) = A_1(\zeta - a_1)^{-1} + \bar{A}_1(\zeta - 1/\bar{a}_1)^{-1} + B_1(\zeta - a_2)^{-1} + \bar{B}_1(\zeta - 1/\bar{a}_2)^{-1}$$
$$= (A_1 + \bar{A}_1 + B_1 + \bar{B}_1)\zeta^{-1} + 0(\zeta^{-2}).$$

Now, by the condition

$$\operatorname{Re} A_1 = -\operatorname{Re} B_1$$

we see that the coefficient of ζ^{-1} vanishes. Hence,

$$F'(\zeta) = C\zeta^{-2} + 0(\zeta^{-3}), \tag{4.3}$$

at infinity, where C is finite or zero. Thus, the two terms in dz of (3.6) have the following behaviour:

$$\left.\begin{aligned} g(\zeta)(\zeta - a_1)^{-1+1/n}(\zeta - a_2)^{-1-1/n} &\sim g(\infty)\zeta^{-2}, \\ F'^{\beta}(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n}(\zeta - a_2)^{1+1/n} &\sim C^{\beta}\{g(\infty)^{-1}\}\zeta^{-2} \end{aligned}\right\} \tag{4.4}$$

Thus, the residue is zero, and hence a closed path is described in the z-plane as we describe a closed path in the ζ-plane at large distances. In fact, as $\zeta \to \infty$, z does not become infinite. The point at infinity in the ζ-plane is mapped into some point at finite distances from the airfoils.

The description of the mapping of such paths thus demonstrate that (3.6) is the desired mapping for the circular lattice of n airfoils. Two remarks are to be added. (1) The point $\zeta = a_2$ maps into the point at infinity in the z-plane as one can easily see from (3.6). (2) Without the second term of (3.6), as it is indeed the case for incompressible flows, the point $\zeta = a_1$ maps into a finite point in the z-plane, which may be conveniently taken to be the origin. In the com-

pressible case, one must exclude a certain region in the ζ-plane where condition (d) is not satisfied. Thus, as one approaches the boundary of this region from $\zeta = a_2$, the point z moves inwards from infinity to the image of this boundary in the z-plane. When the boundary is crossed, the path in the z-plane reverses its direction of motion, and we have the outer region covered again. This is the familiar phenomenon associated with a vanishing Jacobian, or specifically, the limiting line in compressible flow.

The geometrical part of the proof is now complete.

(β) *Satisfaction of the differential equation.* Having clarified the geometry of the mapping, we want to show now that (3.7) and (3.8) give the velocity potential, the stream function and the magnitude and direction of velocity of the compressible flow in the z-plane. We shall first show that the differential equations for ϕ and ψ are satisfied, leaving the discussion of the boundary conditions to the next section.

It is well known that under the approximation (3.4), the relations

$$dz = dF/w^* - \tfrac{1}{4}\,\overline{w^*\,dF} \qquad F = F(w^*) \tag{4.5}$$

with z, F, w^* defined by

$$\left.\begin{aligned} z &= x + iy, \qquad F = \phi + i\psi \\ w^* &= \frac{2q}{1 + \sqrt{1+q^2}}\, e^{-i\theta} \end{aligned}\right\} \tag{4.6}$$

would give a solution of the *differential* equations of compressible flow. Instead of trying to establish a relation between F and w^* directly, we introduce an *auxiliary* variable ζ such that $F(\zeta)$ and $w^*(\zeta)$ are analytic functions. This is done in (3.6) by taking $F(\zeta)$ to be given by (3.5) and

$$w^*(\zeta) = F'(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n}(\zeta - a_2)^{1+1/n}, \tag{4.7}$$

which is indeed (3.8). Thus, *differential* relations are satisfied, and ϕ, ψ, q, θ as calculated indeed define a field of compressible flow.

(γ) *Satisfaction of conditions in the large.*

The essential conditions to be satisfied in the large are the following:

(i) The flow field is invariant, when rotated through an angle of $2\pi/n$;

(ii) A contour such as $C_1'C_2' \cdots C_{n-1}'C_1'$ should give a certain amount of circulation and enclose a certain source strength;

(iii) A contour drawn to enclose all the airfoils has a similar property; and

(iv) Each airfoil should be a stream line, around which the circulation is equal to one-nth the difference of those in (ii) and (iii).

It is not difficult at all to verify that these conditions are satisfied. The first point follows directly from the geometry of the mapping. The flow field in the ζ-plane is mapped onto $1/n$ of the z-plane corresponding to one determination of the n-th roots in (3.6). The other portions of the z-plane are obtained by rotating the first portion over angles of $2\pi/n$, $4\pi/n$, etc., and corresponds to the other determinations of the n-th roots in (3.6). The stream line patterns are identical,

because $F(\zeta)$ is merely changed by additive constants when the angles of $\zeta - a_1$ and $\zeta - a_2$ change by 2π.

To verify the second point, let us recall that the contour $C_1'C_2' \cdots C_{n-1}'C_1'$ corresponds to the contour CD around $\zeta = a_1$ described n times. It is therefore obvious that the source strength and the circulation are given by the real and imaginary parts of $n\bar{A}_1$.

A contour drawn to enclose all the airfoils corresponds to the contour (iii) of Fig. 3 described n times. The source strength and the circulation are given by the real and imaginary parts of $-n\bar{B}_1$. The negative sign comes from the fact that we are mapping $\zeta = a_2$ to $z = \infty$, and therefore the positive sense of the contour has been reversed. Notice that

$$Re(n\bar{A}_1) = Re(-n\bar{B}_1),$$

by the original condition imposed on A_1 and B_1. The difference of circulation is

$$\mathrm{Im}(-n\bar{B}_1 - n\bar{A}_1) = n\,\mathrm{Im}(A_1 + B_1).$$

The airfoils are stream lines, because ψ is constant along each of them. The circulation around each of them is the same as that around the circle in the ζ-plane, which is $\mathrm{Im}(A_1 + B_1)$. This is exactly $1/n$ the circulation difference just discussed.

The proof is now complete for the compressible flow through a circular lattice.

*Simplification to Straight Lattice.** When $n \to \infty$, the mapping function (3.6.) becomes

$$dz = g(\zeta)(\zeta - a_1)^{-1}(\zeta - a_2)^{-1}d\zeta - \tfrac{1}{4}\overline{F'^2(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)(\zeta - a_2)d\zeta}. \tag{4.8}$$

Thus, the branch point nature of the singularity at $\zeta = a_1$ and $\zeta = a_2$ disappears and a path around $\zeta = a_1$ and $\zeta = a_2$ changes z by a constant value without changing the mapping function. Thus, a translation in the z plane is produced, which may be regarded as the limit of the previous rotation through an angle $2\pi/n$, as n goes to infinity. The lattice may be regarded as arranged on a circle of infinite radius.

Both the points $\zeta = a_1$ and $\zeta = a_2$ are mapped to infinity, the first to the left of the lattice, the second to the right, if the value of $g(\zeta)$ is properly chosen at $\zeta = a_1$. The velocity of flow at $z = \mp\infty$ are respectively given by (cf. (3.8)).

$$\frac{2q}{1 + \sqrt{1 + q^2}}\, e^{-i\theta} = \frac{A_1}{g(a_1)}\,(a_1 - a_2), \tag{4.9}$$

and

$$\frac{2q}{1 + \sqrt{1 + q^2}}\, e^{-i\theta} = \frac{B_1}{g(a_2)}\,(a_2 - a_1). \tag{4.10}$$

* In a yet unpublished paper, Prof. L. Bers showed that the integral equation method developed by him (Ref. 3) for the case of a single profile can be extended to the case of a straight cascade.

These formulae give the change both in the magnitude and in the direction of the velocity when the fluid passes through the lattice.

Incompressible fluid. If the fluid is incompressible, the second terms on the right-hand sides of (3.6) and (4.8) are to be dropped, leaving a conformal mapping between the z-plane and the ζ-plane. The mapping covers the *whole* of the z-plane up to the center of symmetry. One can see this by noting that the Jacobian (4.1) will no longer vanish on account of the second factor (which is now identically unity).

In the case of a straight lattice, condition (c) of §3 leads to

$$g(a_1) = g(a_2). \tag{4.11}$$

The velocities of flow at infinity are respectively

$$qe^{-i\theta} = \frac{A_1}{g(a_1)}(a_1 - a_2) \qquad z = -\infty$$

$$qe^{-i\theta} = \frac{B_1}{g(a_2)}(a_2 - a_1) \qquad z = +\infty$$

The change of velocity is given by $-B_1/A_1$. A similar factor for change can be obtained from (4.9) and (4.10).

5. Choice of the function $g(\zeta)$. A few words about the choice of the function $g(\zeta)$ is in order. A simple function regular outside the unit circle is

$$g(\zeta) = \left(1 - \frac{\lambda_1}{\zeta}\right)\left(1 - \frac{\lambda_2}{\zeta}\right)\cdots\left(1 - \frac{\lambda_n}{\zeta}\right). \tag{5.1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are points *inside* the circle. The mapping is completely regular on the circle, and the body produced would be void of a sharp trailing edge, which is needed for an airfoil. By making one of the points, say λ_1, to lie on the circle, a cusp will be produced, provided

$$F'(\lambda_1) = 0$$

so that the Jacobian (4.1) does not become infinite through the second factor. Otherwise, the Jacobian changes its sign in the neighborhood of $\zeta = \lambda_1$ outside the circle, and the whole mapping procedure is made useless. If a finite angle is desired, one may choose, say,

$$g(\zeta) = \left(1 - \frac{\lambda_1}{\zeta}\right)^{1-\epsilon}\left(1 - \frac{\lambda_2}{\zeta}\right)^{1+\epsilon}\cdots\left(1 - \frac{\lambda_n}{\zeta}\right)$$

where $\epsilon > 0$. The angle would then be $\epsilon\pi$. The power $1 + \epsilon$ is necessary for some factor in order that the function $g(\zeta)$ be single-valued outside of $|\zeta| = 1$.

It is however difficult to know in advance whether the mapping will be multiply covered; i.e., whether the contour of the airfoil will be a curve with double points. It seems that experience should help in such a problem. So far the author has not encountered such difficulties.

Page 127: In the last formula omit the exponent 1+ϵ. Also omit the statement concerning this exponent which follows.

6. Discussion. The previous treatment gives only the basis of the theory. It should not be difficult to work out the aerodynamic properties of the lattice in terms of the function $g(\zeta)$ analogous to the general airfoil theory. Thus, it might be expected that $g(a_1)$, $g'(a_1)$, $\cdots$ and $g(a_2)$, $g'(a_2)$, $\cdots$ would determine many of the aerodynamic characteristics. From these discussions, design procedures are to be evolved.

It is also noteworthy that the present formulation (as well as the compressible flow past a single airfoil) is actually independent of the hodograph plane, although the historical development has been that way. The essential step is a mapping

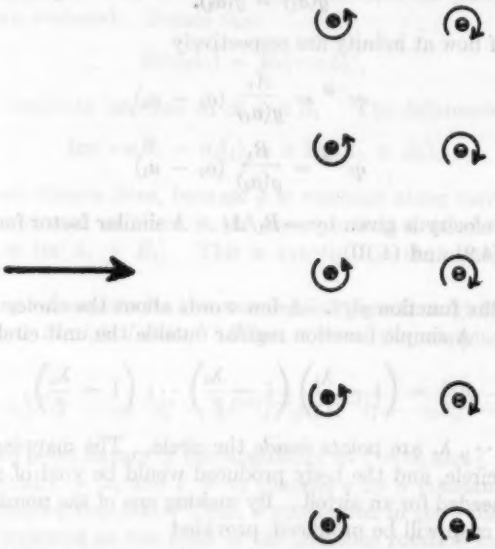

FIG. 4. GENERATION OF A STRAIGHT LATTICE BY SUPERPOSING A UNIFORM FLOW WITH A SYSTEM OF COMPLEX SOURCES.

of an incompressible flow into a compressible flow. More precisely, mappings of the kind (3.6), which depends explicitly on $F'(\zeta)$, make $F(\zeta)$ at once the complex velocity potential of an incompressible flow in the ζ-plane and of a compressible flow in the z-plane with velocity given by (3.8). In general, one may consider

$$\left.\begin{aligned} dz &= k(\zeta)\,d\zeta - \tfrac{1}{4}\overline{[F'^2(\zeta)][k(\zeta)]^{-1}\,d\zeta}, \qquad \oint dz = 0, \\ \phi + i\psi &= F(\zeta), \qquad \frac{2qe^{-i\theta}}{1+\sqrt{1+q^2}} = \frac{F'(\zeta)}{k(\zeta)}, \end{aligned}\right\} \tag{6.1}$$

to be mapping an incompressible flow in the ζ-plane into a compressible flow in the z-plane. If $k(\zeta)$ is not much different from unity, the two flows will be similar

in nature. The reduction of (6.1) to conformal mapping of incompressible flows is very easy.

This immediately suggests other methods of obtaining flows of the types discussed above. One can easily construct a *particular* incompressible flow of the general nature desired. Two infinite rows of source-vortex combinations of equal and opposite source strength superposed with a uniform velocity will produce an incompressible flow past a straight lattice (Fig. 4). Similarly, the

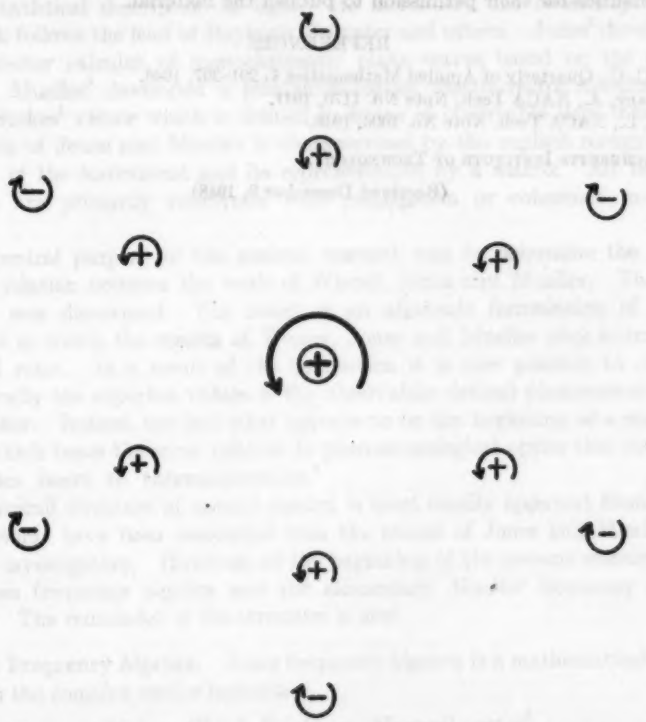

FIG. 5. GENERATION OF A CIRCULAR LATTICE BY SUPERPOSING A SYSTEM OF COMPLEX SOURCES.

circular arrangement of such pairs with a complex source at the center will produce the flow past a circular lattice (Fig. 5). The shape of the bodies thus obtained may be not very desirable. Nevertheless, one can make them the basis for producing desired flows by mappings of the kind (6.1) just described.

Doubtless, this basic incompressible flow can be mapped into the incompressible flow outside a unit circle with certain singularities in the field, such as that given by (3.6), with the second term on the right dropped. The direct mapping of the final compressible flow from the plane of the circle will then be of the kind (3.6). Only experience will tell whether these alternative methods just proposed

will be more advantageous than the general theory for certain types of design problems. It is probably safe to say that at least the general theory of §3 will be easier to apply for the determination of the general flow characteristics.

Acknowledgment. This work was carried out at Brown University and sponsored by Pratt & Whitney Aircraft Division of United Aircraft Corporation. The author is indebted to these organizations and the Navy Department Bureau of Aeronautics for their permission to publish the material.

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(Received December 9, 1948)

OPTICAL ALGEBRA*

By Nathan Grier Parke III

There have been three major lines of attack on the problem of obtaining a useful and satisfying mathematical representation of light and the instruments through which it passses. Wiener[1] developed a generalized harmonic analysis and a statistical description of light. He introduced the coherency matrix. His work follows the lead of Rayleigh, Schuster and others. Jones[2] developed a matrix-vector calculus of monochromatic plane waves based on the electric vector. Mueller[3] developed a phenomenological matrix-vector calculus based on the Stokes[4] vector which is defined in terms of observable light intensities. The work of Jones and Mueller is characterized by the explicit recognition of the role of the instrument and its representation by a matrix. All these researches are primarily concerned with polarization or coherence, not with images.

The central purpose of the present research was to determine the mathematical relation between the work of Wiener, Jones and Mueller. The interrelation was discovered. The result is an algebraic formulation of optical statistics in which the results of Wiener, Jones and Mueller play natural and essential roles. As a result of the connection it is now possible to compute theoretically the expected values of the observables defined phenomenologically by Mueller. Indeed, one has what appears to be the beginning of a statistical optics which bears the same relation to phenomenological optics that statistical mechanics bears to thermodynamics.[4]

The overall structure of optical algebra is most readily apparent from Fig. 1. The algebras have been associated with the names of Jones and Mueller, the original investigators. However, at the beginning of the present research, only the Jones frequency algebra and the elementary Mueller frequency algebra existed. The remainder of the structure is new.

Jones Frequency Algebra. Jones frequency algebra is a mathematical system in which the complex vector function

$$E(\omega) = (F_1(\omega), F_2(\omega)) \ .=. \text{ Maxwell vector} \tag{1}$$

* This work has been supported in part by the Signal Corps, the Air Materiel Command and O.N.R. It is a condensation of the mathematical part of a thesis, "Matrix Optics," submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology (1948).

[1] Wiener, N.: Acta Math. **55**: 117 (1930); Extensive bibliography.

[2] Jones, R. C.: Jour. Opt. Soc. Am., **31**: 488 (1941); **37**: 707 (1947); **38**: 671 (1948).

[3] Mueller, H.: M.I.T., course 8.262; Fall 1946, and an unpublished manuscript.

[4] The subject is treated in full in: Parke, N. G.: "Matrix Optics," Ph.D. Thesis, Course VIII, M.I.T., 1948. 200 pp. Ch. I, Generalized Optical Algebra; Ch. II, Illustrations and Examples; Ch. III, Statistical Harmonic Analysis; Ch. IV, Quasi-Stationary Scattering. The present paper summarizes the mathematical aspects of the algebraic structure. Space limitations preclude any motivation or interpretation. A slightly more extensive summary will be found in Report No. 70, Research Laboratory of Electronics, M.I.T., 1948.

[5] The symbol ".=." is read "called" or "represents."

characterizes the radiation. The complex matrix function

(2) $$J(\omega) = [J_{ij}(\omega)] \text{ .=. Jones matrix}$$

characterizes the instrument. The definition of J is the basic transfer relation,

(3) $$E' = 2\pi JE, \text{ for all } E; E' \text{ .=. outgoing radiation.}$$

The following operations occur,

(4) $$E = E_1 + E_2 \text{ .=. vector addition .=. coherent superposition}$$

			ELEMENTARY ALGEBRAS	GENERALIZED ALGEBRAS
JONES ALGEBRAS	Theoretical Quantities	ω-Algebra	Jones Frequency Algebra Eq. (3) $E' = 2\pi JE$	Symbols Maxwell Quantities .=. E, e, f .=. Radiation Stokes Quantities .=. L, ϕ, S .=. Radiation Jones Quantities .=. J, j .=. Instrument Mueller Quantities .=. M, m .=. Instrument
		t-Algebra	Jones Time Algebra Eq. (25) $e'(t) = \int_{-\infty}^{\infty} j(t_0)e(t-t_0)\,dt_0$ Eq. (34) $f'_\beta(t) = j^\alpha_\beta(t_0)f_\alpha(t-t_0)$	Generalized Jones Time Algebra Eq. (30) $f'^{i}_{\alpha}(t) = j^{i\beta}_{\alpha}(t_0)f^{j}_{\beta}(t-t_0)$
			Wiener's Generalized Harmonic Analysis	
MUELLER ALGEBRAS	Observables	t-Algebra	Mueller Time Algebra Eq. (38) $\varphi'_{\alpha\beta}(t) = m^{\gamma\delta}_{\alpha\beta}(t_1,t_2)\varphi_{\gamma\delta}(t-t_1+t_2)$	Generalized Mueller Time Algebra Eq. (50) $\varphi'_{\alpha\beta}(t) = m^{jk\gamma\delta}_{\alpha\beta}(t_1,t_2)\varphi_{jk\gamma\delta}(t-t_1+t_2)$
		ω-Algebra	Mueller Frequency Algebra Eq. (40) $S'_{\alpha\beta}(\omega) = M^{\gamma\delta}_{\alpha\beta}(\omega)S_{\gamma\delta}(\omega)$	Generalized Mueller Frequency Algebra Eq. (52) $S'_{\alpha\beta}(\omega) = M^{jk\gamma\delta}_{\alpha\beta}(\omega)S_{jk\gamma\delta}(\omega)$

FIG. 1. THE STRUCTURE OF OPTICAL ALGEBRA

(5) $$J = J_2J_1 \text{ .=. matrix multiplication .=. instruments in series}$$

(6) $$J = J_1 + J_2 \text{ .=. instruments in parallel.}$$

Mueller Frequency Algebra. Mueller frequency algebra is a mathematical system in which the complex vector function

(7) $$L(\omega) = (I(\omega), M(\omega), C(\omega), S(\omega)) \text{ .=. Stokes vector}$$

characterizes the radiation. The complex matrix function

(8) $$M(\omega) = [M_{ij}(\omega)] \text{ .=. Mueller matrix}$$

characterizes the instrument. The definition of M is the basic transfer relation,

(9) $$L' = ML, \text{ for all } L; L' \text{ .=. outgoing radiation.}$$

The following operations occur,

(10) $L = L_1 + L_2$.=. vector addition .=. incoherent superposition

(11) $M = M_2M_1$.=. matrix multiplication .=. instruments in series

(12) $M = M_1 + M_2$.=. matrix addition .=. instruments in parallel.

Wiener's Generalized Harmonic Analysis. Wiener has pointed out that, in the electromagnetic theory of light, the field vectors E and B are not observables as they appear to be at lower frequencies.[6] Optical observations always end in intensity measurements using the eye, a photographic plate, the photoelectric effect, a bolometer, etc. The quantities of the Maxwell theory most nearly corresponding with these observations are,

$$\tfrac{1}{2}(E.D + B.H) \text{ .=. energy density}$$

$$E \times H \text{ .=. Poynting vector .=. energy flow}$$

quantities depending "quadratically" on the field quantities E, B, D, H. This linear-quadratic duality is exactly the relation between Jones and Mueller algebra. The correspondence is supplied by Wiener's generalized harmonic analysis.

As introduced by Wiener, the generalized harmonic analysis of a function $f(t)$, takes place in two steps,

(13)
$$\varphi(t) = \operatorname*{Lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t + t_0)f^*(t_0)\, dt_0$$
$$= \langle f(t + t_0)f^*(t_0)\rangle_{\text{Avg.}} \text{ .=. auto-correlation}$$

(14)
$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t)e^{i\omega t}\, dt$$
$$= F\varphi(t) \text{ .=. Fourier transformation.}$$

If this is applied to

(15) $f(t) = a_n f_n(t)$; a_n complex; (Summation convention),

then

(16) $\varphi(t) = a_n\varphi_{nm}(t)a_m^*$

where

(17) $\varphi_{nm}(t) = \langle f_n(t + t_0)f_m^*(t_0)\rangle$.=. Interference matrix.

The Fourier transformation of eq. (16) yields,

(18) $S(\omega) = a_n S_{nm}(\omega)a_m^*$

where

(19) $S_{nm}(\omega) = F\varphi_{nm}(t)$.=. Spectral matrix.

[6] Wiener, N.: Jour. Franklin Institute **207**: 525 (1929).

Wiener prefers the matrix

$$C_{nm}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{nm}(t)\,\frac{e^{i\omega t}-1}{it}\,dt \;.=.\; \text{coherency matrix.} \tag{20}$$

When differentiation is meaningful,

$$S_{nm}(\omega) = (d/d\omega)C_{nm}(\omega). \tag{21}$$

For a monochromatic plane wave of frequency ω, it turns out that the spectral matrix is

$$S_{nm}(\omega) = \frac{1}{2}\begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix} \tag{22}$$

δ .=. unit impulse function

(I, M, C, S) .=. monochromatic Stokes vector.

Jones Time Algebra. Jones time algebra is the more fundamental. Its Fourier transform (when defined on a finite time interval) is Jones frequency algebra. Briefly,

$e(t) = (f_1(t), f_2(t))$.=. Maxwell vector .=. radiation, (23)

$$E(\omega) = Fe(t), \tag{24}$$

$$e'(t) = \int_{-\infty}^{\infty} j(t_0)e(t-t_0)\,dt_0 \;.=.\; \text{convolution} \;.=.\; \text{transfer relation}, \tag{25}$$

$$E'(\omega) = Fe'(t) \tag{26}$$

$$J(\omega) = Fj(t) \tag{27}$$

$$e(t) = e_1(t) + e_2(t) \;.=.\; \text{vector addition} \;.=.\; \text{superposition}. \tag{28}$$

J is the response of an optical instrument to polarized monochromatic waves. j is the response of an optical instrument to polarized impulses. The correlation applies to functions of time; hence the introduction of Jones time algebra. The quantum theory of scattering, or the Mie theory of scattering, refers to monochromatic plane waves and gives the J of Jones frequency algebra. It turns out that it is only necessary to make conceptual use of Jones time algebra. The J's can be used directly to compute the M's of Mueller frequency algebra, e.g., eq. (53).

Generalized Jones Algebra. The problem which motivates the generalization of Jones algebra is scattering from a statistical assemblage of centers, Fig. 2. The following notation is useful,

$f_\beta^j(t)$.=. set of N incoming wavelets; $j = 1, 2, \cdots, N$

$j_\alpha^{i\beta}(t)$.=. set of M scattering centers; $i = 1, 2, \cdots, M$

These multi-index quantities are direct sums of the corresponding elementary quantities. Using the summation convention one has,

$$f'^{ij}_{\alpha}(t) = \int_{-\infty}^{\infty} j^{i\beta}_{\alpha}(t_0) f^{j}_{\beta}(t - t_0)\, dt_0 \;.=.\; MN \text{ outgoing wavelets.} \tag{29}$$

The last operation is a combined Kronecker product, summation and convolution. In order to free the calculation of integral signs one introduces the "convolution convention": *Repetition of a time variable with a single bar under both occurrences implies convolution.* Thus,

$$f'^{ij}_{\alpha}(t) = j^{i\beta}_{\alpha}(\underline{t_0}) f^{j}_{\beta}(t - \underline{t_0}). \tag{30}$$

If only a single wave comes in,

$$f'^{i}_{\alpha}(t) = j^{i\beta}_{\alpha}(t_0) f_{\beta}(t - \underline{t_0}). \tag{31}$$

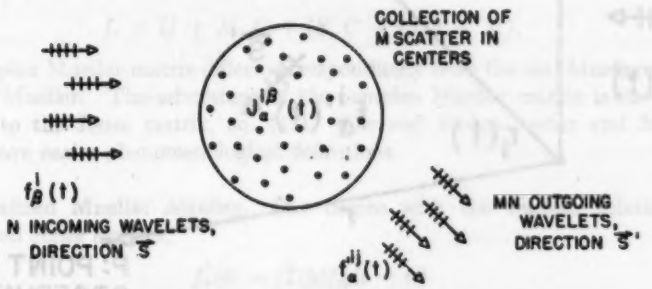

FIG. 2. SCATTERING, A BASIC OPTICAL PATTERN

The geometry of the relative retardations of the various scattered wavelets $f'^{i}_{\alpha}(t)$ must be considered before summation on i is carried out. Let $r_i(t)$ be the position of the M scattering centers, Fig. 3.
At the point of observation P, eq. (31) becomes

$$f'^{i}_{\alpha}\left(t - \frac{s' \cdot r}{v}\right) = j^{i\beta}_{\alpha}(\underline{t_0}) f_{\beta}\left(t - \frac{s' \cdot (r - r_i(\bar{t}))}{v} - \frac{s \cdot r_i(\bar{t})}{v} - \underline{t_0}\right), \tag{32}$$

i not summed, v .=. velocity of radiation. After change of epoch, eq. (32) becomes,

$$f'_{\alpha}(t) = j^{i\beta}_{\alpha}(t_0) f_{\beta}\left(t - \frac{(s - s') \cdot r_i(t - \underline{t_0})}{v}\right); \qquad (i \text{ summed}). \tag{33}$$

This result involves a quasi-stationary assumption on the scattering centers but leads to many interesting results in agreement with a widely scattered literature, including scattering of x-rays by crystals, liquids and gases. Elementary Jones algebra could not describe depolarizing instruments and partially polarized radiation, the generalized algebra can.

Elementary Mueller Algebra. The relation between Jones and Mueller algebra is most easily understood if one restricts attention to the effect of a single elementary instrument,

$$f'_\beta(t) = j^\alpha_\beta(t_0) f_\alpha(t - t_0). \tag{34}$$

The Wiener correlation of the incoming radiation is

$$\varphi_{\alpha\beta}(t) = \langle f_\alpha(t + t_0) f^*_\beta(t_0) \rangle. \tag{35}$$

One can introduce a "correlation convention": *Repetition of a time variable with a double bar under both occurrences implies correlation*, and rewrite eq. (35),

$$\varphi_{\alpha\beta}(t) = f_\alpha(t + \underline{t_0}) f^*_\beta(\underline{t_0}). \tag{36}$$

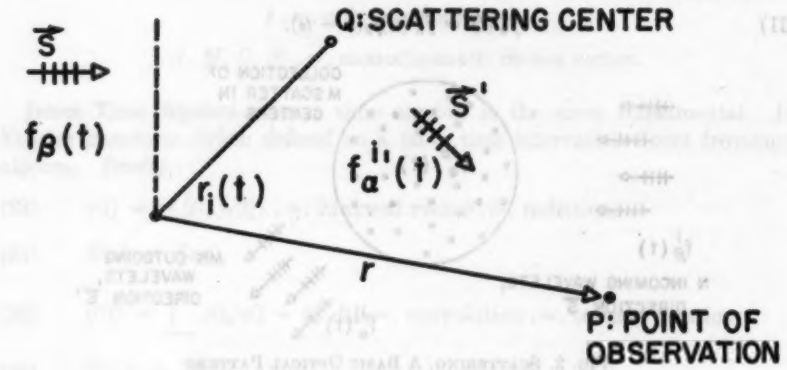

FIG. 3. RETARDATION GEOMETRY

Also,

$$\varphi'_{\alpha\beta}(t) = f'_\alpha(t + \underline{t_0}) f'^*_\beta(\underline{t_0}) = j^\gamma_\alpha(t_1) j^{\delta *}_\beta(t_2) f_\gamma(t - \underline{t_1} + \underline{t_2} + \underline{t_0}) f^*_\delta(\underline{t_0}). \tag{37}$$

From this one obtains the important formula,

$$\varphi'_{\alpha\beta}(t) = m^{\gamma\delta}_{\alpha\beta}(t_1, \underline{t_2}) \varphi_{\gamma\delta}(t - t_1 + \underline{t_2}) \quad .=. \text{ transfer relation}, \tag{38}$$

in which

$$m^{\gamma\delta}_{\alpha\beta}(t_1, t_2) = j^\gamma_\alpha(t_1) j^{\delta *}_\beta(t_2) \quad .=. \text{ Mueller function}, \tag{39}$$

the Kronecker product of two Jones matrices. The Fourier transform of eq. (39) gives

$$S'_{\alpha\beta}(\omega) = M^{\gamma\delta}_{\alpha\beta}(\omega) S_{\gamma\delta}(\omega) \tag{40}$$

where,

$$M^{\gamma\delta}_{\alpha\beta}(\omega) = (2\pi J^\gamma_\delta(\omega))(2\pi J^\delta_\beta(\omega))^* \quad .=. \text{ Mueller function}. \tag{41}$$

If one introduces the "index reduction transformation,"

(42) $$i = \alpha + 2(\beta - 1); j = \gamma + 2(\delta - 1)$$

one obtains,

(43) $M_{\alpha\beta}^{\gamma\delta}(\omega) = M_{ij}$.=. complex Mueller matrix,

(44) $S_{\gamma\delta}(\omega) = L_j$.=. complex Stokes vector,

(45) $S'_{\alpha\beta}(\omega) = L'_i$.=. complex Stokes vector.

On dropping indices and using matrix notation,

(46) $$L'(\omega) = M(\omega)L(\omega).$$

These matrices and vectors differ from those used by Mueller. In particular, the complex Stokes vector turns out to be,

(47) $$L = (I + M, C + iS, C - iS, I - M).$$

The complex Mueller matrix differs correspondingly from the real Mueller matrix used by Mueller. The advantage of the complex Mueller matrix is its simple relation to the Jones matrix, eq. (41). The real Stokes vector and Mueller matrix have easier phenomenological definitions.

Generalized Mueller Algebra. One begins with the transfer relation for generalized Jones algebra,

(33) $$f'_\alpha(t) = j_\alpha^{\beta\delta}(\underline{t_0})f_{j\beta}(t - \underline{t_0})$$

where,

(48) $$f_{j\beta}(t) = f_\beta\left(t - \frac{(s - s')\cdot r_j(t)}{v}\right).$$

Wiener correlation gives,

(49) $$\varphi'_{\alpha\beta}(t) = [j_\alpha^{i\gamma}(\underline{t_1})j_\beta^{k\delta *}(\underline{t_2})]f_{i\gamma}(t + \underline{t_0} - \underline{t_1})f_{k\delta}^*(\underline{t_0} - \underline{t_2}).$$

This reduces to,

(50) $$\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{ik\gamma\delta}(\underline{t_1}, \underline{t_2})\varphi_{ik\gamma\delta}(t - \underline{t_1} + \underline{t_2}),$$

where

(51) $m_{\alpha\beta}^{ik\gamma\delta}(t_1, t_2) = j_\alpha^{i\gamma}(t_1)j_\beta^{k\delta *}(t_2)$.=. Mueller function.

The Fourier transform of eq. (5) yields,

(52) $$S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{ik\gamma\delta}(\omega)S_{ik\gamma\delta}(\omega),$$

where,

(53) $M_{\alpha\beta}^{ik\gamma\delta}(\omega) = (2\pi J_\alpha^{i\gamma}(\omega))(2\pi J_\beta^{k\delta}(\omega))^*$.=. Mueller function.

When $j = k$, the M's are called auto-functions, when $j \neq k$ they are called cross-functions.

The difficulties of the scattering problem are reduced to the evaluation of the interference,

$$\varphi_{jk\gamma\delta}(\omega) = f_{j\gamma}(l + \underline{w})f^*_{k\delta}(\underline{w}). \tag{54}$$

For an important class of cases, the Fourier transform of eq. (54) separates, i.e.,

$$S_{jk\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega). \tag{55}$$

This leads to a simplification in which $N_{jk}(\omega)$ is characteristic of the statistical distribution of scattering centers. In this case,

$$M^{\gamma\delta}_{\alpha\beta}(\omega) = N_{jk}(\omega)M^{jk\gamma\delta}_{\alpha\beta}(\omega), \tag{56}$$

a general law for the composition of Mueller matrices, for instruments in parallel. Each scattering center is thought of as an elementary instrument.

The Form of $N_{jk}(\omega)$ in Typical Cases. The importance of $N_{jk}(\omega)$ is well illustrated by three examples which embody a unification of the treatment of scattering from liquids, solids and gases, i.e., coherent scattering (solids), partially coherent scatterings (liquids, electrolytes), and incoherent scattering (gases).

1. Coherent Scattering,

$$N_{jk}(\omega) = e^{i(s-s')\cdot(r_j-r_k)\omega/v}. \tag{57}$$

2. Incoherent Scattering,

$$N_{jk} = \delta_{jk}. \tag{58}$$

3. Partially Coherent Scattering (Radial Symmetry),

$$N_{jk}(s) = \begin{cases} 1, & \text{when } j = k \\ \dfrac{i(s)}{nV} & \text{when } j \neq k, \end{cases} \tag{59}$$

$$s = \frac{\omega\,|\,s - s'\,|}{v}; \quad i(s) = F(\rho(r) - n) \tag{60}$$

n .=. concentration

$\rho(r)$.=. radial concentration density.

4. Special Case of an Electrolyte

$$i(s) = \pm \frac{1}{2(1 + (s/\kappa)^2)} \tag{61}$$

$+$.=. unlike-signed ions

$-$.=. like-signed ions

$$\kappa^2 = \frac{2nZ^2\,|\,e\,|}{\epsilon kT} \tag{62}$$

$N_{jk}(\omega)$ is called a Debye statistical matrix because it is closely related to some of Debye's results in electrolyte theory and in x-ray scattering.

Composition of Mueller Matrices. Written in full, eq. (54) becomes,

$$\varphi_{jk\gamma\delta}(t) = f_\gamma\left(t + \underline{t_0} - \frac{(s - s')\cdot r_j(t - \underline{t_0})}{v}\right) f_\delta^*\left(\underline{t_0} - \frac{(s - s')\cdot r_k(\underline{t_0})}{v}\right). \tag{63}$$

Making the quasi-stationary approximation,

$$\begin{aligned}\varphi_{jk\gamma\delta}(t) &\overset{\text{Def}}{=} f_\gamma\left(t + \underline{t_1} - \frac{(s - s')\cdot(r_j(\underline{t_2}) - r_k(\underline{t_2}))}{v}\right) f_\delta^*(\underline{t_1}) \\ &= \varphi_{\gamma\delta}\left(t - \frac{(s - s')\cdot(r_j(\underline{t_2}) - r_k(\underline{t_2}))}{v}\right).\end{aligned} \tag{64}$$

The Fourier transform of the last expression gives,

$$S_{jk\gamma\delta}(\omega) = S_{\gamma\delta}(\omega)e^{i\omega(s-s')\cdot(r_j(\underline{t_2})-r_k(\underline{t_2}))/v} \tag{65}$$

If one defines,

$$N_{jk}(\omega) = \operatorname*{Lim}_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i\omega(s-s')\cdot(r_j(t)-r_k(t))/v}\,dt \tag{66}$$

then

$$S_{ij\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega) \tag{67}$$

which is eq. (55), as required. Under these circumstances one has a general rule for the composition of Mueller matrices, eq. (56).

Acknowledgement. I am indebted to Hans Mueller for pointing out the problem, guiding the research and encouraging an independent investigation of the mathematical aspects. I am also indebted to Y. W. Lee for helpful discussions of the close relation between my results and his results in network theory. Lee uses a system matrix analogous to the Jones matrix, and a squared system matrix corresponding to the Mueller matrix. This type of analysis appears to be applicable to any linear system with a transfer function and a significant linear-quadratic duality such as voltage-power.

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(Received November 15, 1948)

DESIGN OF A CIRCUIT TO APPROXIMATE A PRESCRIBED AMPLITUDE AND PHASE[1]

By R. M. Redheffer

1. Introduction. The system function of a network is a complex function, and hence has a real part, an imaginary part, an amplitude, and a phase. If any one of these four "parts" of the function (as we may call them) is specified over the whole frequency range, then the other three parts are automatically determined. Or again if any two parts are specified over a finite frequency range, no matter how small, then all four parts are determined over the whole frequency range. These and similar results follow from the fact that the system function is an analytic function of the complex frequency.

Suppose, then, that we are given two arbitrary curves, to be matched respectively by two parts of the system function. If we succeed in matching one of these prescribed curves over the whole frequency range, we cannot hope, in general, to come anywhere near the other; for no freedom remains to make adjustments of any kind. Alternatively, if we succeed in matching both curves over some small frequency interval, we cannot expect to match them, even approximately, outside of that range; for again no freedom of adjustment remains.

In practical work, nevertheless, one often requires a network for which the real and imaginary parts, or the amplitude and phase, do conform to prescribed functions. Examples of problems leading to such requirements are the broadbanding problem, and certain problems connected with feedback circuits in servos or amplifiers. It is natural, therefore, to inquire whether two parts of the system function can be prescribed at least approximately over a frequency range. The chief problems are to find some systematic method of minimizing the error, and to estimate the minimum error so obtained. Such is the purpose of the present note. The subject and its applications were suggested by E. A. Guillemin, who also supplied all results taken from circuit theory.

By the transformation $\omega = \tan \phi/2$ the frequency domain ω is transformed to a corresponding ϕ domain, the range $0 \leq \omega < \infty$ going into $0 \leq \phi < \pi$. In the ϕ domain we may confine our attention to this range of ϕ, and hence all functions may be assumed even or odd at will, and may be supposed without loss of generality to have period 2π. The amplitude $A(\phi)$ may then be written as a Fourier cosine series,

$$A(\phi) = a_0 + a_1 \cos \phi + a_2 \cos 2\phi + \cdots . \tag{1}$$

As noted above, the phase $-B(\phi)$ is completely determined by such a specification of amplitude. With the present choice of variable, which is used by N.

[1] This work was supported in part by the Signal Corps, the Air Materiel Command, and O.N.R.

Wiener, the specification of phase has a particularly simple form, namely,

$$B(\phi) = a_1 \sin \phi + a_2 \sin 2\phi + \cdots \tag{2}$$

with the coefficients a_k the same as those in Eq. (1). This equality of coefficients represents the Wiener condition for physical realizability of the two functions $A(\phi)$ and $B(\phi)$. The phase is taken as $-B(\phi)$ and below as $-b(\phi)$ rather than as $B(\phi)$ or $b(\phi)$, incidentally, to give the theory a more symmetric form.

The functions $A(\phi)$ and $B(\phi)$ represent the actual output of the network, while $a(\phi)$ and $b(\phi)$ represent the desired output. Our task, then is to investigate the approximation of two arbitrary functions $a(\phi)$, $b(\phi)$ by Fourier cosine and sine series, respectively, the same coefficients being used in both cases. The results to be described are obtained with this formulation of the problem as starting point, and the approximation is always specified in the mean-square sense; that is, the coefficients are so determined that the integral

$$\begin{aligned} I &= \int \left| a(\phi) + ib(\phi) - \sum_0^n a_k e^{ik\phi} \right|^2 d\phi \\ &\equiv \int \left[a(\phi) - \sum_0^n a_k \cos k\phi \right]^2 d\phi + \int \left[b(\phi) - \sum_0^n a_k \sin k\phi \right]^2 d\phi \end{aligned} \tag{3}$$

is minimized. Our attention will be confined to the coefficients a_k ; once these are known, the remainder of the design leads to problems of circuit theory with which we shall not be concerned here.

2. Approximation Over Whole Frequency Range Without Constraint. To approximate $a(\phi)$ and $b(\phi)$ over the whole frequency range, which corresponds to limits of integration $0 - \pi$ in Eq. (3), one should determine the coefficients a_k by the equations

$$a_k = (1/\pi) \int_0^\pi [a(\phi) \cos k\phi + b(\phi) \sin k\phi] \, d\phi. \tag{4}$$

The Fourier cosine coefficients of $a(\phi)$ would be the optimum values for a_k if we were interested in approximating $a(\phi)$ alone, while the Fourier sine coefficients of $b(\phi)$ would be optimum for $b(\phi)$ alone. This is well known from the theory of Fourier series. The above result, then, says that the optimum coefficients for simultaneous approximation are the arithmetic means of the optimum coefficients for individual approximation.

3. Approximation Over Finite Frequency Range Without Constraint. It may be shown[2] that the error in the above calculation will not usually tend to zero as n tends to infinity; in other words an arbitrarily good approximation over the

[2] See M.I.T. Research Laboratory of Electronics, Technical Report No. 54, for proof of this and other results of the text. Copies of this report may be obtained from the Document Room, R.L.E., M.I.T.

whole frequency range cannot be obtained, no matter how many terms are used in the approximating series. If on the other hand we attempt to obtain a good approximation over only part of the frequency range, with no restriction on the behavior of the approximating functions outside of that range, then the error can always be made arbitrarily small. Because of this property, which has also been proved, the question of approximation over a finite interval is worth considering in detail.

The optimum coefficients are determined as follows, for a range $0 - s$ in the ϕ domain with $s \leqq \pi$. First compute a set of numbers P_{jk} given by

$$P_{jk} = [\sin (j - k)s]/(j - k) \quad (k \neq j); \quad P_{jj} = s. \tag{5}$$

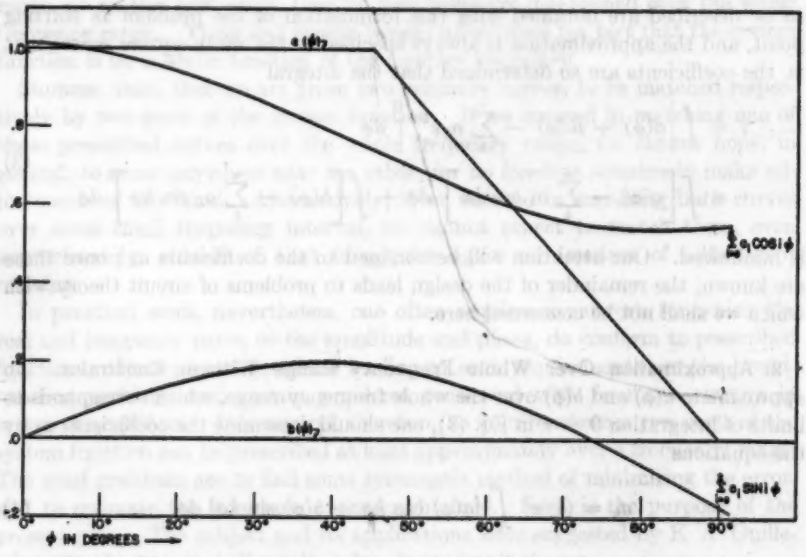

FIG. 1. Mean square approximation of $a(\phi)$ and $b(\phi)$, with $n = 2$, $s = \pi/2$

These depend only on s, the range of approximation, and not on n or the functions $a(\phi)$, $b(\phi)$ being approximated. For routine calculations, therefore, they would be computed once for all, and would be available in the form of a table.

Next compute a set of constants A_k given by the formula

$$A_k = \int_0^s [a(\phi) \cos k\phi) + b(\phi) \sin k\phi] \, d\phi \tag{6}$$

which depends on the functions being approximated. The optimum coefficients are obtained by solving the following set of linear equations:

$$\sum_{j=0}^{n} P_{jk} a_j = A_k \quad (j = 0, 1, 2, \cdots n). \tag{7}$$

Graphical examples of the results obtained by this procedure are given in Figs 1–3.

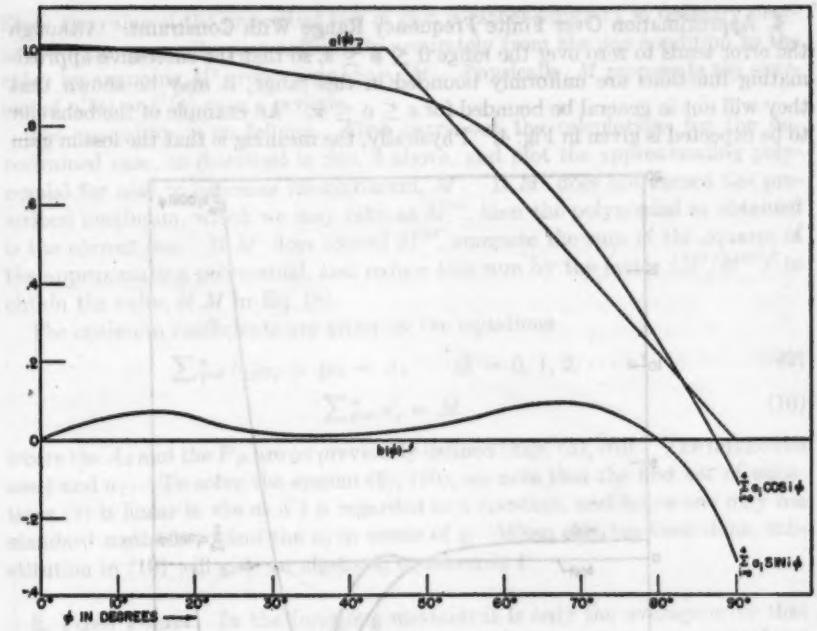

FIG. 2. Same, $n = 4$

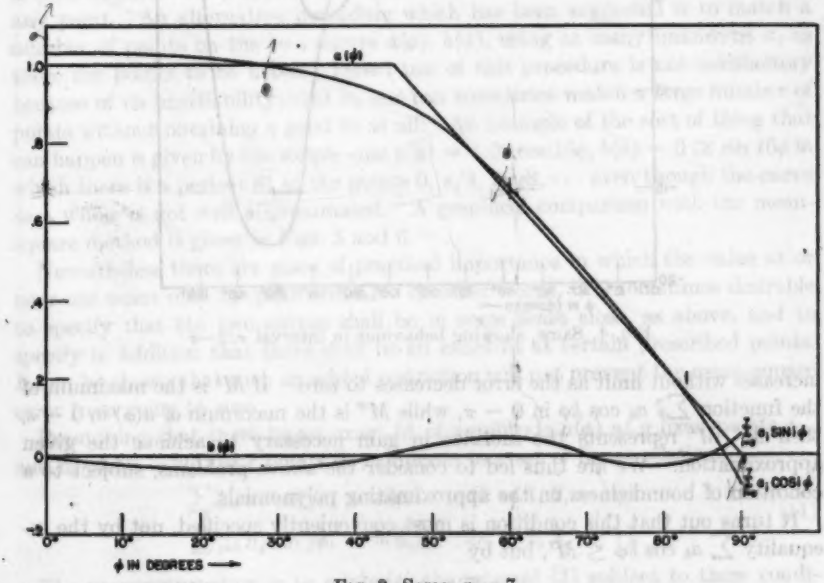

FIG. 3. Same, $n = 7$

4. Approximation Over Finite Frequency Range With Constraint. Although the error tends to zero over the range $0 \leq \phi \leq s$, so that the successive approximating functions are uniformly bounded in this range, it may be shown that they will not in general be bounded for $s \leq \phi \leq \pi$. An example of the behavior to be expected is given in Fig. 4. Physically, the meaning is that the loss in gain

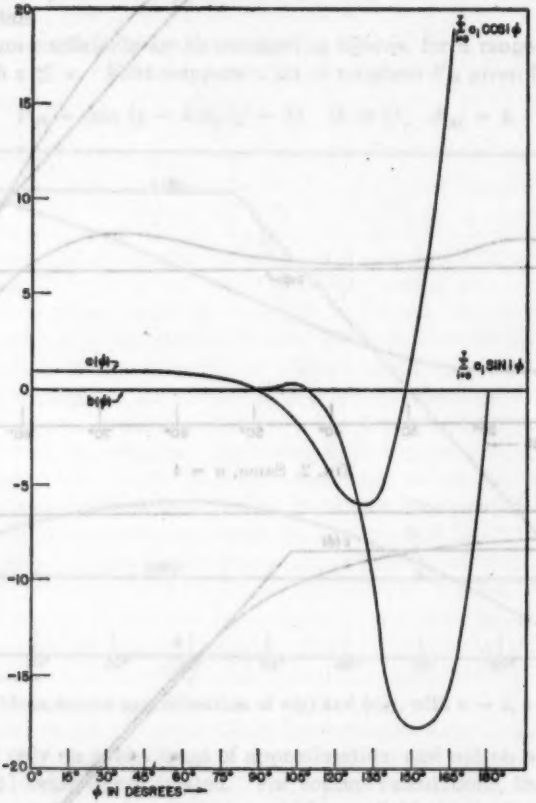

FIG. 4. Same, showing behaviour in interval $\pi/2$—π

increases without limit as the error decreases to zero. If M' is the maximum of the function $\sum_0^n a_k \cos k\phi$ in $0 - \pi$, while M'' is the maximum of $a(\phi)$ in $0 - s$, then M''/M' represents the sacrifice in gain necessary to achieve the given approximation. We are thus led to consider the above problems, subject to a condition of boundedness on the approximating polynomials.

It turns out that this condition is most conveniently specified, not by the inequality $\sum a_k \cos k\phi \leq M'$, but by

$$\sum_0^n a_k^2 \leq M. \tag{8}$$

Since the value of the prescribed loss of gain is rather arbitrary in ordinary practice, we can generally pass sufficiently accurately from the one condition to the other by assuming M' proportional to $\sqrt{M}$. Physically, M represents the average of $a^2(\phi)$ or $b^2(\phi)$ over a period.

The procedure is as follows. First carry out the calculations for the unrestrained case, as described in Sec. 3 above, and plot the approximating polynomial for $a(\phi)$ to estimate its maximum, M'. If M' does not exceed the prescribed maximum, which we may take as M''', then the polynomial so obtained is the correct one. If M' does exceed M''', compute the sum of the squares of the approximating polynomial, and reduce this sum by the factor $(M'/M''')^2$ to obtain the value of M in Eq. (8).

The optimum coefficients are given by the equations

$$\sum_{j=0}^{n} P_{jk} a_j + \xi a_k = A_k \qquad (k = 0, 1, 2, \cdots n) \tag{9}$$

$$\sum_{j=0}^{n} a_j^2 = M \tag{10}$$

where the A_k and the P_{jk} are as previously defined [Eqs. (5), (6)]. The unknowns are ξ and a_j. To solve the system (9), (10), we note that the first set of equations (9) is linear in the a_j if ξ is regarded as a constant, and hence one may use standard methods to find the a_j in terms of ξ. When this has been done, substitution in (10) will give an algebraic equation in ξ.

5. Fixed Points. In the foregoing methods it is only the average error that is considered, and the approximating polynomials may not give an exact fit at any point. An alternative procedure which has been suggested is to match a number of points on the two curves $a(\phi)$, $b(\phi)$, using as many unknowns a_j as there are points to be fitted. Direct use of this procedure is not satisfactory because of its unreliability; that is, one can sometimes match a large number of points without obtaining a good fit at all. An example of the sort of thing that can happen is given by the simple case $a(\phi) = 1 \cong \cos 16\phi$, $b(\phi) = 0 \cong \sin 16\phi$ in which there is a perfect fit at the points $0, \pi/8, 2\pi/8, \cdots$ even though the curve as a whole is not well approximated. A graphical comparison with the mean-square method is given in Figs. 5 and 6.

Nevertheless there are cases of practical importance in which the value at or near one point may be quite critical. In other words, it is sometimes desirable to specify that the two curves shall be in some sense close, as above, and to specify in addition that there shall be an exact fit at certain prescribed points. It can be shown that such an added restriction will not prevent the mean-square error from going to zero.

Prescribing that there be an exact fit of amplitude $a(\phi)$ at q fixed points ϕ_k, and of phase $-b(\phi)$ at r points $\bar{\phi}_k$, amounts to a set of linear relations

$$\begin{aligned} \sum_{j=0}^{n} a_j \cos j\phi_k &= a(\phi_k) \qquad (k = 1, 2, \cdots q) \\ \sum_{j=0}^{n} a_j \sin j\bar{\phi}_k &= b(\bar{\phi}_k) \qquad (k = 1, 2, \cdots r). \end{aligned} \tag{11}$$

The proper procedure is to minimize the integral (3) subject to these condi-

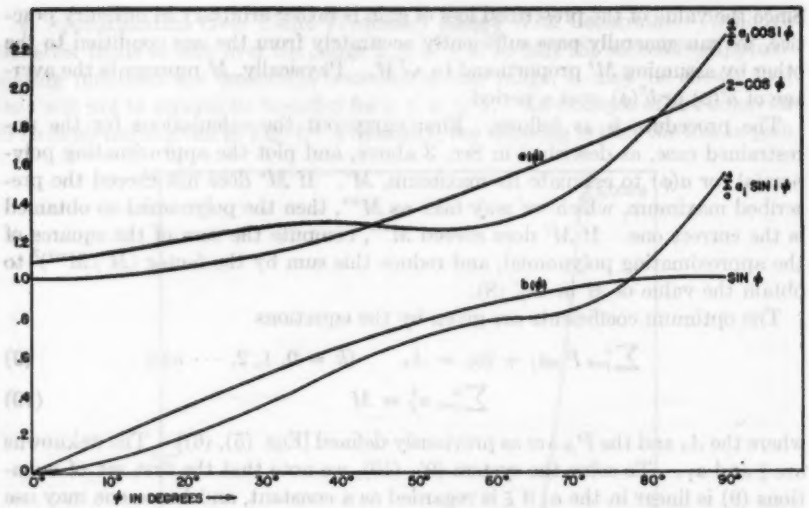

FIG. 5. Mean square approximation, $a(\phi) = 2 - \cos\phi$, $b(\phi) = \sin\phi$; $n = 4$, $s = \pi/2$

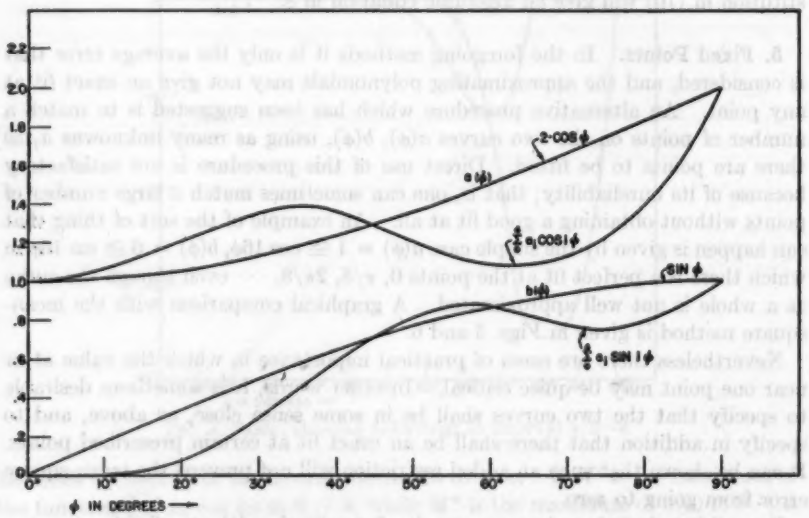

FIG. 6. Same functions and range, points matched at 0°, 45°, 90°

tions. For simplicity we shall assume that only one point is prescribed, though the theory is probably not much different for the general case. If

$$\sum p_j a_j = p \tag{12}$$

is the linear relation analogous to (11), then the optimum coefficients a_k are

obtained from (12) and the equations

$$\sum_{j=0}^{n} (P_{jk} - P_{jm} p_k/p_m) = A_k - A_m p_k/p_m \quad (k = 0, 1, \cdots n) \qquad (13)$$

which are also linear, and may be solved in the same way as the system (7). The coefficient p_m is any coefficient in (12) which is not zero; the P_{jk} and A_k have the usual meaning (Eqs. (5), (6)).

6. Other Methods. In conclusion we note a few alterations that could have been made in the general procedure hitherto followed. Instead of minimizing the expression (3), for example, one could treat the corresponding expression with the second integral multiplied by some constant λ. In this way the relative accuracy of the two approximations for $a(\phi)$ and $b(\phi)$ could be changed. A large value of λ will tend to favor $b(\phi)$, so that it is accurately approximated, while a small value will weight the error in $a(\phi)$ the more heavily. It is probable that this generalization of the problem would lead to no new complications, though we shall not carry out the calculations here.

A second, more radical, departure from our procedure would be to work in the impedance plane rather than in the ϕ-plane. The condition for physical realizability would then be essentially that the Fourier transforms of the two functions be equal, and we should be led to a minimizing problem subject to this as a constraint.

Finally, for the partial interval, which is perhaps the most interesting case in practice, one could begin by constructing a set of polynomials $\sum_0^n g_{nk} \cos k\phi$, $\sum_0^n h_{nk} \sin k\phi$ orthogonal over the fundamental interval $0 - s$ rather than over $0 - \pi$. The treatment of the error would then be much simpler, and would proceed much as in the $0 - \pi$ case treated above. But this increased simplicity would be compensated by an increased complexity of the condition for physical realizability: instead of having mere equality of the coefficients, one would have a set of linear equations.

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(Received March 1, 1948)

A THEOREM ON COLORING THE LINES OF A NETWORK

By Claude E. Shannon

A topological coloring problem has been suggested in connection with the color coding of wires in electrical units such as relay panels. In these units there are a number of relays, switches, and other devices $A, B, \cdots, E$ to be interconnected. The connecting wires are first formed in a cable with the wires to be connected to A coming out at one point, those to B at another, etc., and it is necessary, in order to distinguish the different wires, that all those coming out of the cable at the same point be differently colored. There may be any number of wires joining the same two points but no wire joins more than two points. Assuming that not more than m wires end at any one point, the question arises as to the least number of different color codings that is sufficient for any network.

Theorem: The lines of any network can be colored so that no two lines with a common junction have the same color using at most $[\frac{3}{2}m]$ colors, where m is the maximum number of lines touching one junction. This number of colors is necessary for some networks.

Simple networks requiring $[\frac{3}{2}m]$ colors can be constructed as follows. For $m = 2n$, let each pair of the three junctions A, B, C be connected with n lines. Since all the lines must obviously be different, $3n = [\frac{3}{2}m]$ colors are necessary. For $m = 2n + 1$, connect AB with n lines, BC with n lines and AC with $n + 1$ lines. Here again all lines must be different and there are $3n + 1 = [\frac{3}{2}(2n + 1)] = [\frac{3}{2}m]$ lines. Another example for $m = 3$ is furnished by the cross connection of two pentagons *abcde* and *ABCDE* by lines *aA*, *bD*, *cB*, *dE*, *eC*.

For the sufficiency proof let us first suppose m even. Now if N is our given network it is well known that we may add lines and junction points to get a regular network N' of degree m, i.e., one in which exactly m lines end at each junction point. If we can color N' we can surely color N. A theorem due to Peterson states that any regular network of even degree $m = 2n$ can be factored into n regular second degree graphs. In our case let the factors of N' be $N_1, N_2, \cdots, N_n$. Each of these is merely a collection of polygons which do not touch each other, and each N_i, therefore, can be colored with at most three colors. This gives a total of $3n = \frac{3}{2}m$ colors.*

Peterson has conjectured that any regular bridgeless network of odd degree $2n + 1$ can be factored into one first and n second degree graphs, and if this is true the theorem follows easily in the odd case. However this conjecture has never been proved for $m > 3$ and we will use a different attack.

The theorem will be proved by induction, making the coloring of N depend on coloring a network with one less junction point. Let us eliminate from N one junction point P and the $m = 2n + 1$ lines coming to it and assume the remaining network to be satisfactorily colored with $3n + 1$ colors. Let the junctions that were connected to P in the original network be numbered $1, 2, \cdots, s$, and

* This proof for m even was suggested by R. M. Foster.

suppose there were p_1 parallel lines in the first group G_1 connecting P to junction 1, etc. Now after coloring the reduced network we have left available at junction 1 at least $[(3n + 1) - (2n + 1 - p_1)] = n + p_1$ colors, at junction 2, at least $(n + p_2)$ colors, etc. By choosing properly from these available colors and by suitable interchanges of certain colors in the part of the network already colored we will show that the lines from P can be satisfactorily colored.

Let us arrange the data in a table as follows.

		Colors									
		1	2	3	·	·	·	·	·	·	$(3n + 1)$
	1	1	1	0	1	·	·	·	·	·	·
	2	1	0	1	0	·	·	·	·	·	·
	·										
lines	·	·			·			·			·
	·	·			·			·			·
	$(2n + 1)$	·			·			·			·

FIG. 1

In this array the $2n + 1$ lines from P are listed vertically, the $3n + 1$ colors horizontally. If a certain color is available for a certain line a 1 is placed at the intersection of the corresponding row and column, otherwise a 0. In a row corresponding to a line in G_i there will be $(n + p_i)$ 1's. By the use of three operations on this array we will arrange to obtain a series of 1's along the main diagonal and this will represent a coloring scheme for the network. These operations are:

1. Interchange of columns. This corresponds to merely renumbering the colors.

2. Interchange of rows. This corresponds to renumbering the lines from P.

3. Interchange of colors in a chain of two colors. Two junctions will be said to be chained together for two colors if we can go from one junction to the other along lines which alternate these colors. If we have a satisfactorily colored network and interchange the two colors in a chain of that network over its entire length (note that in a correctly colored network a chain cannot branch out) then it is clear that the network will still be satisfactorily colored. We will also use the fact that if only one of the two colors in question appears at each of three distinct junctions then only one pair of the junctions (at most) can be chained together for these two colors, since a chain can only have two ends. Now consider the array of Fig. 1. Suppose that in the first row there is a 0 in one column and a 1 in another. Interchanging these two colors in the chain starting from the junction that line one is connected to will be seen to be equivalent to interchanging these two columns for this row and all other rows that are chained to it.

Let us suppose that we have arranged to get 1's on the main diagonal D down to a certain point. We will show that we can get another 1 on the diagonal.

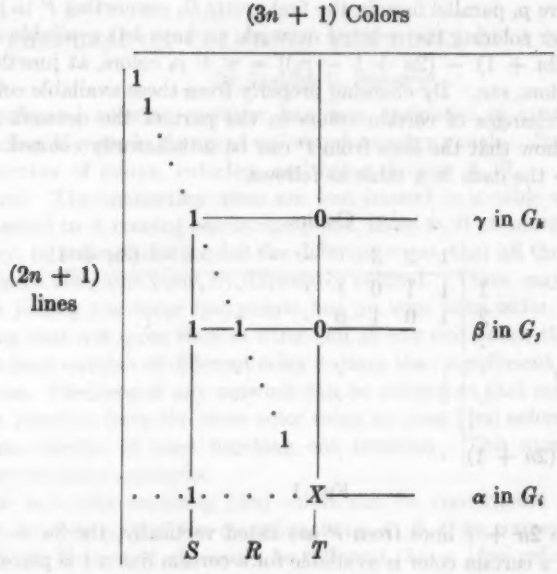

FIG. 2

Referring to Fig. 2, if there are any 1's in the next line α, on or to the right of T, one of these may be brought by an interchange of columns, to position X. Assuming this is not true there will be $n + p_i$ 1's to the left of column T in α (assuming α is in G_i). Hence there are $n + p_i$ rows above α having a 1 in D in the same column as a 1 in α. At least $n + 1$ of these rows are not in G_i, since G_i has p_i members and we have accounted for one already, namely α. Let β be one of these, belonging, say, to G_j. If β has a 1 on or to the right of T, by an interchange first of columns, then of α and β this may be moved to X without affecting the 1's along D. Assuming this is not true, there are $n + p_j$ 1's on β to the left of T and hence $n + p_j$ rows above α have a 1 in D in the same column as a 1 in β, and of these at least n do not belong to G_j (as it only has p_j members). Now there are not more than $2n$ rows above α and therefore the n rows we have associated with β and the $n + 1$ we have associated with α must have at least one in common, i.e., there exists a row not belonging to G_i or G_j and having a 1 in D in the same column that α and β also have a 1. Call this row γ, and suppose it belongs to G_k. If γ has a 1 on or to the right of T it may be moved to X by first interchanging columns and then rows α and γ as before. Assuming this is not true, there are 0's and 1's at the intersections of α, β, γ, and T, S, as shown in Fig. 2 and X is 0. Hence at least one of α, β, γ is not chained to either of the others by the two colors of T and S. If it is α, interchange the chain starting at i and the 1 at the αS intersection moves to X without affecting D. If it is β,

interchange the chain starting at j and then rows α and β. This moves the 1 at the βS intersection to X and its place is taken by the 1 at αS. If it is γ, interchange the chain starting at k and then the rows α and γ so that the 1 at γS moves to X and the 1 at αR takes its place.

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(Received September 14, 1948)

THE DISTRIBUTION OF SINGULAR SHOCK DIRECTIONS*

By T. Y. Thomas

1. Introduction. The following paper contains an extension of the results of several articles, recently written, which deal with shock waves attached to the vertex of a pointed obstacle in a uniform field of flow. We are concerned with the plane steady supersonic flow of gases without the usual assumption that the flow behind the shock wave is irrotational and isentropic, although viscosity and thermal conductivity are taken to be zero. The figure shows the upper half of the shock line and illustrates the situation assumed. The velocity of the flow before the shock line (incident flow) is of constant magnitude w and is directed along the positive x^1 axis; the coordinate system is rectangular and has its origin at the vertex V of the obstacle. Angles α and ω give the inclination of the shock line and obstacle respectively at V, ν is the unit normal vector to the shock line directed into the region behind this line and λ denotes the unit tangent vector to the shock line. The directions of λ and ν are such that the vector pair λ, ν has the same orientation as the coordinate axes.

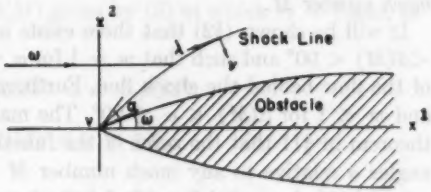

FIG. 1

Denote by s the arc length along the shock line and by η the arc length along the obstacle measured in the direction of flow; observe here that s is measured along the shock line from a point on this line toward the vertex V in conformity with the direction of the tangent vector λ. Now put $K_{(n)} = d^nK/d\eta^n$ and $\kappa_{(n)} = d^n\kappa/ds^n$ for $n = 0, 1, 2, 3, \ldots$, where K and κ denote the curvatures of the obstacle and shock line respectively. Then $K_{(0)} = K$ and $\kappa_{(0)} = \kappa$. It has been shown that there exist relations of the form[1]

$$K_{(n)} = G_n(M, \alpha)\kappa_{(n)} + H_n, \qquad n = 0, 1, 2, 3, \ldots, \tag{1}$$

at point V, where M is the mach number of the incident flow and the H_n are polynomials in κ and its derivatives with respect to s of order less than n; moreover the quantities $G_n(M, \alpha)$ and the coefficients of the polynomials H_n are rational functions of M^2 with coefficients depending on the inclination α of the shock line. Actually the above relations hold at any point of the shock line (as well as all following results) when α is the inclination of the shock line and K the curvature of the stream line behind the shock line at this point, but we shall be concerned here primarily with the relations (1) at the vertex V of the obstacle. *For a given value of $M > 1$, a value of α for which any one of the functions $G_n(M, \alpha) = 0$ is said to be singular relative to M.* When α is singular there will exist in general

* Prepared under Navy Contract N6onr-180, Task Order V, with Indiana University.

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no shock wave solution* to the hydrodynamical problem under the assumptions of continuity and differentiability involved in the derivation of the relation (1) for which the coefficient $G_n(M, \alpha)$ vanishes. It is evident, therefore, that these singular values have an important role in the theory of shock waves attached to the vertices of curved obstacles.

When the condition is imposed that the shock wave is a compression shock the range of permissible values of α is given by[2]

$$\alpha_0(M) < \alpha \leqq 90°, \tag{2}$$

where $\alpha_0(M)$ is the value of $\sin^{-1} 1/M$ in the first quadrant. Graphs of the functions $-G_0(M, \alpha)$ and $G_1(M, \alpha)$ have been constructed in preceeding papers (*loc. cit.* 1 and 2) for selected values of M and α in the above range (2) and the zeros of the functions G_0 and G_1 have bee noted. *It is the object of this paper to determine the distribution of the zeros of all functions* $G_n(M, \alpha)$, $n = 0, 1, 2, \ldots$, *or in other words the singular values of* α *within the range* (2) *for any specified mach number* M.

It will be shown (§2) that there exists a number $\beta = \beta(M)$ such that $\alpha_0(M) < \beta(M) < 90°$ and such that $m = 1$ for $\alpha = \beta(M)$ where m is the mach number of the flow behind the shock line. Furthermore $m > 1$ for $\alpha_0(M) < \alpha < \beta(M)$ and $m < 1$ for $\beta(M) < \alpha \leqq 90°$. The main result of the paper consists of the theorem in §11 that the zeros of the functions $G_n(M, \alpha)$, i.e. the singular shock angles α relative to any mach number $M > 1$, are dense in the interval $\beta(M) < \alpha < 90°$. A general formula for the functions $G_n(M, \alpha)$ in case $m \geqq 1$ can be derived from the results of §7 and preceeding sections. For $m < 1$ the explicit formula for these functions is given in §10.

2. Graph of the equation $m^2 = 1$. Under the assumptions of the problem we know that

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{v^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{w^2}{2},$$

where p, ρ and v are the pressure, density and magnitude of the velocity at any point behind the shock line and p_1, ρ_1 and w are the constant values of the pressure, density and velocity in the incident flow; also γ is the ratio of the two specific heats c_p and c_v. Now $c^2 = \gamma p/\rho$ and $c_1^2 = \gamma p_1/\rho_1$. Also by definition the mach numbers M and m for the flow before and behind the shock line are given by $M = w/c_1$ and $m = v/c$. Making these substitutions the above equation can be written in the form

$$m^2 = \frac{[(\gamma - 1)M^2 + 2]}{\gamma - 1}\frac{c_1^2}{c^2} - \frac{2}{\gamma - 1}. \tag{3}$$

But $c_1^2/c^2 = (\rho/\rho_1)/(p/p_1)$ and[2]

$$\frac{\rho}{\rho_1} = \frac{2(M^2 \sin^2 \alpha - 1)}{(\gamma - 1)M^2 \sin^2 \alpha + 2} + 1,$$

$$\frac{p}{p_1} = \frac{2\gamma}{\gamma + 1}(M^2 \sin^2 \alpha - 1) + 1.$$

* An exceptional case is furnished by the ordinary shock solution for flow past a wedge.

Hence equation (3) becomes

$$(4)\quad m^2 = \frac{1}{\gamma - 1}\left\{\frac{(\gamma+1)^2[(\gamma-1)M^2+2]M^2\sin^2\alpha}{[(\gamma-1)M^2\sin^2\alpha+2][2\gamma M^2\sin^2\alpha-(\gamma-1)]} - 2\right\}.$$

Now put $m = 1$ and $\alpha = \beta$ in equation (4); solving the resulting equation for $\sin\beta$ we find

$$(5)\quad \sin^2\beta = \frac{1}{4\gamma M^2}\left\{[(\gamma+1)M^2+(\gamma-3)] + \sqrt{[(\gamma+1)M^2+(\gamma-3)]^2+16\gamma}\right\}.$$

Hence, since $\sin\beta$ is positive in the interval $\alpha_0(M) < \beta \leqq 90°$, equation (4) admits at most one solution $\alpha = \beta(M)$ for which $m = 1$ in the interval $\alpha_0(M) < \alpha \leqq 90°$. But for $M > 1$ we readily obtain from (4) that $m > 1$ when $\alpha = \alpha_0(M)$, i.e. $\sin\alpha = 1/M$. Also we find from (4) that $m < 1$ when $\alpha = 90°$. Hence there exists, in the above interval, a value $\beta(M)$ given by (5) at which $m = 1$ and, as stated in §1, it follows that

$$m > 1, \quad \text{for } \alpha_0(M) < \alpha < \beta(M),$$

$$m < 1, \quad \text{for } \beta(M) < \alpha \leqq 90°.$$

Values of the function $\beta(M)$ have been calculated for specified values of $M > 1$ and the results entered in the accompanying table which also contains the values of $\alpha_0(M)$; in these calculations we have taken $\gamma = 1.405$. The graphs of the functions $\alpha_0(M)$ and $\beta(M)$ have been constructed from the entries of this table. By drawing a line parallel to the α-axis, corresponding to a specified value of M, and then dropping perpendiculars from its points of intersection with the curves $\alpha_0(M)$ and $\beta(M)$ we obtain the interval $\alpha_0 < \alpha \leqq 90°$ on the α-axis and the location of the point $\alpha = \beta(M)$ in this interval.

3. Simplification of the problem by rotation of coordinates. Consider the formula[3]

$$(6)\quad K = \frac{u_{\alpha,\beta}\, w_\alpha u_\beta}{v^3},$$

for the curvature K of any stream line behind the shock line; in the right member of (6) the u_β are the components of the velocity vector, the w_α are the components of a vector perpendicular to the velocity, of the same magnitude as the velocity vector, and such that the vectors u, w have the orientation of the coordinate axes*. We thus have $u_1 = w_2$ and $u_2 = -w_1$. Now differentiate (6) repeatedly

* The summation convention applies. However in this equation and those which follow we have put all indices in the covariant position (which is strictly legitimate since rectangular systems are employed) and thus have reserved the upper or contravariant position for exponents in the usual sense.

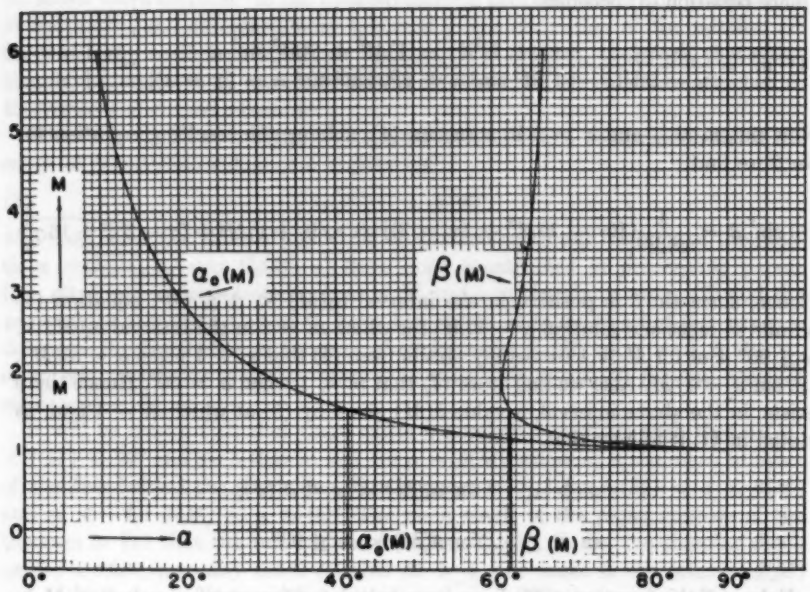

FIG. 2

M	α_0 (deg.)	β (deg.)
1.00	90.00	90.00
1.05	72.25	77.63
1.10	65.38	73.20
1.30	50.28	65.05
1.50	41.82	62.23
1.75	34.85	61.27
2.00	30.00	61.45
2.25	26.38	61.97
2.50	23.57	62.58
3.00	19.48	63.70
3.50	16.60	64.53
4.00	14.48	65.18
4.50	12.85	65.63
5.00	11.55	66.00
5.50	10.47	66.25
6.00	9.60	66.45

with respect to the arc length η along the stream line to obtain equations of the form

$$K_{(1)} = \frac{u_{\alpha,\beta\gamma}\, w_\alpha\, u_\beta\, u_\gamma}{v^4} + \cdots,$$

. . . . . .

. . . . . .

. . . . . .

$$K_{(n)} = \frac{u_{\alpha,\beta\gamma\cdots\delta}\, w_\alpha\, u_\beta\, u_\gamma \cdots u_\delta}{v^{n+3}} + \cdots,$$

where the "dotes" denote terms in the derivatives of the components u_α of lower order than those which appear explicitly in the first term of any equation. We note here that there are n indices in the set $\gamma \cdots \delta$ (where $n = 0, 1, 2, \cdots$) corresponding to the subscript n appearing in the symbol $K_{(n)}$ in the last of the above equations.

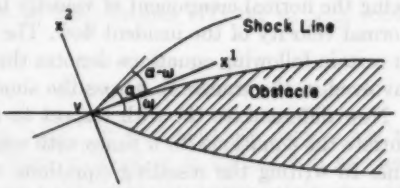

FIG. 3

Now rotate the coordinate axes through the angle ω so that the new x^1 axis is tangent to the obstacle at the vertex V. Our equations are invariant under this rotation and for simplicity we shall continue to use the designations for quantities previously employed. Relative to the new system we have, at V, the following relations: $u_1 = v$, $u_2 = 0$ and $w_1 = 0$, $w_2 = v$. Hence the above equation for $K_{(n)}$ becomes

$$K_{(n)} = \frac{u_{2,11\cdots1}}{v} + \cdots. \tag{7}$$

Now the first term in the right member of (1) originates entirely from the first term in the right member of (7). *Hence the problem of determining the coefficient* $G_n(M, \alpha)$ *in* (1) *is essentially the problem of determining the quantity* $u_{2,11\cdots1}$ *relative to the rotated axes.*

4. The algebraic conditions. The differential equations governing the flow behind the shock line are given, under our hypotheses, by

$$\begin{cases} p_{,\alpha} + \rho u_\sigma u_{\alpha,\sigma} = 0, \\ \rho_{,\sigma} u_\sigma + \rho u_{\sigma,\sigma} = 0, \\ \rho u_{\sigma,\tau} u_\sigma u_\tau - \gamma p u_{\sigma,\sigma} = 0. \end{cases} \tag{8}$$

In addition there are certain relations holding immediately behind the shock line, namely

$$\begin{cases} \mu_{\alpha,\beta} \lambda_\beta = A_\alpha \kappa, \\ p_{,\beta} \lambda_\beta = B\kappa, \\ \rho_{,\beta} \lambda_\beta = C\kappa, \end{cases} \tag{9}$$

where the A_α are the components of a vector and the B and C are scalars. The explicit formulas of A_α, B and C are*

$$A_\alpha = -\frac{B\nu_\alpha}{\rho_1 u_{1n}} + \frac{[\rho]}{\rho}(u_{1n}\lambda_\alpha - u_t\nu_\alpha),$$

$$B = -\frac{4\rho u_n u_t}{\gamma+1},$$

$$C = \frac{2u_t}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\},$$

where the subscript 1 indicates that the quantity in question is considered for the uniform flow before the shock line; the symbols u_t and u_n or u_{1n} are the components of velocity tangential and normal to the shock line, the quantity u_n being the normal component of velocity behind the line and the u_{1n} denoting the normal velocity of the incident flow. The bracket [ ] in the above equations and in certain following equations denotes the jump or discontinuity in the quantity involved in the transition across the shock line.

Now differentiate (8) with respect to the variables $x^\beta \cdots x^\gamma$ and also differentiate the equations (9) n times with respect to the arc length s along the shock line. In writing the resulting equations we shall omit terms not of the highest order in the derivatives of the u_α with respect to the coordinates and also terms of order less than n in derivatives of κ with respect to s since such terms do not enter in the determination of the functions $G_n(M, \alpha)$. We thus obtain

$$p_{,\alpha\beta\cdots\gamma} + \rho u_\sigma u_{\alpha,\sigma\beta\cdots\gamma} = 0, \tag{10}$$

$$\rho_{,\sigma\beta\cdots\gamma}u_\sigma + \rho u_{\sigma,\sigma\beta\cdots\gamma} = 0, \tag{11}$$

$$u_{\sigma,\tau\beta\cdots\gamma}u_\sigma u_\tau - c^2 u_{\sigma,\sigma\beta\cdots\gamma} = 0, \tag{12}$$

where we have put $c^2 = \gamma p/\rho$, and

$$u_{\alpha,\mu\beta\cdots\gamma}\lambda_\mu\lambda_\beta\cdots\lambda_\gamma = A_\alpha\kappa_{(n)}, \tag{13}$$

$$p_{,\mu\beta\cdots\gamma}\lambda_\mu\lambda_\beta\cdots\lambda_\gamma = B\kappa_{(n)}, \tag{14}$$

$$\rho_{,\mu\beta\cdots\gamma}\lambda_\mu\lambda_\beta\cdots\lambda_\gamma = C\kappa_{(n)}. \tag{15}$$

We now refer equations (10), $\cdots$, (15) to the rotated axes introduced in §3. Equation (10) then becomes

$$p_{,\alpha\beta\cdots\gamma} + \rho\overline{u}_{\alpha,1\beta\cdots\gamma} = 0, \tag{16}$$

and, interchanging the indices α and β in these equations, we have

$$\overline{u}_{\alpha,1\beta\cdots\gamma} = \overline{u}_{\beta,1\alpha\cdots\gamma}. \tag{17}$$

* These formulas for the A_α, B and C differ from the formulas for the corresponding quantities in the paper *On Curved Shock Waves* (*loc. cit.* 3) in that they do not involve the factor κ which has been made to appear explicitly in the right members of the equations (9).

Account may be taken of the relations (17) in the choice of independent variables in the process of solving the system (10), $\cdots$, (15). However the problem of determining $u_{2,11\cdots1}$ (§3) reduces to the problem of solving for these quantities within the following system

$$(m^2 - 1)u_{1,1\beta\cdots\gamma} - u_{2,\beta\cdots\gamma} = 0, \tag{18}$$

$$u_{\mu,1\beta\cdots\gamma}\lambda_\mu\lambda_\beta\cdots\lambda_\gamma = -(B/\rho v)\kappa_{(\alpha)}, \tag{19}$$

$$u_{2,\mu\beta\cdots\gamma}\lambda_\mu\lambda_\beta\cdots\lambda_\gamma = A_2\kappa_{(\alpha)}, \tag{20}$$

where $m^2 = v^2/c^2$. Equation (18) results immediately from (12) for the rotated axes, (19) follows from (14) by the substitution (16), and (20) is obtained from (13) by confining our attention to the value 2 of the index α.

Consider the equations (18), (19) and (20) as equations for the determination of the quantities $u_{\mu,\alpha\beta\cdots\gamma}$. But in view of the relations (17) the $u_{\mu,\alpha\beta\cdots\gamma}$ reduce to the following set

$$u_{1,11\cdots1}\,;\quad u_{2,11\cdots1}\,;\quad u_{2,1\cdots12}\,;\quad u_{2,1\cdots122}\,;\quad u_{2,12\cdots2}\,;\quad u_{2,2\cdots2}$$

Observe here that after the first term $u_{1,11\cdots1}$ in this sequence each u has the index 2 in the first position and also that the number of indices 2 in any term (after the first) is one greater than the number in the immediately preceding term. There are $n + 3$ terms $u_{\mu,\alpha\beta\cdots\gamma}$ in this set and there are likewise $n + 3$ equations in the system (18), (19) and (20).

When we order the unknowns $u_{\mu,\alpha\beta\cdots\gamma}$ in the equations (18), (19) and (20) as in the above sequence the determinant of the coefficients of these unknowns will be denoted by Δ. Replacing the second column of Δ by the column consisting of the right members of the system (18), (19) and (20) we obtain a determinant which we shall denote by D. The determinants Δ and D are represented explicitly by Tables B and C, the quantities C_r^{n+1} appearing in these determinants being the ordinary binomial coefficients. Hence, assuming $\Delta \neq 0$, and using Cramer's rule we have

$$u_{2,11\cdots1} = D/\Delta. \tag{21}$$

5. Evaluation of the Δ determinant. For brevity let us put $g = m^2 - 1$ and

$$h_r = C_{r-1}^{n+1}\lambda_1^{n-r+2}\lambda_2^{r-1}.$$

For definiteness let us first suppose that n is odd. Now multiply the next to last column of Δ by g and add to the column two places in advance, i.e. the fourth column counting from right to left. Next multiply the latter column in the determinant, so modified, by g and again add to the column two places in advance, etc. This process will terminate with an addition to the first column of Δ (n odd). Now start with the last column of Δ and operate in a similar manner, i.e. we multiply the last column of Δ by g and add to the column two places in advance, etc. For n odd the process now ends with an addition to the second column of Δ.

m^2-1	0	-1											
	m^2-1	0	-1										
λ_1^{s+1}	$C_1^{s+1}\lambda_1^s\lambda_2$	$C_2^{s+1}\lambda_1^{s-1}\lambda_2^2$	$C_3^{s+1}\lambda_1^{s-2}\lambda_2^3$	$\longrightarrow$	$C_{s-1}^{s+1}\lambda_1^2\lambda_2^{s-1}$	$C_s^{s+1}\lambda_1\lambda_2^s$	λ_2^{s+1}						
	λ_1^{s+1}	$C_1^{s+1}\lambda_1^s\lambda_2$	$C_2^{s+1}\lambda_1^{s-1}\lambda_2^2$	$C_3^{s+1}\lambda_1^{s-2}\lambda_2^3$	$\longrightarrow$	$C_{s-2}^{s+1}\lambda_1^3\lambda_2^{s-2}$	$C_{s-1}^{s+1}\lambda_1^2\lambda_2^{s-1}$	$C_s^{s+1}\lambda_1\lambda_2^s$	λ_2^{s+1}				
				m^2-1	0	m^2-1	0	-1					
			-1		m^2-1	0	-1						
		-1	0										
	-1	0	m^2-1										
m^2-1	0												

$\Delta =$

$$
D = \begin{vmatrix}
m^2-1 & 0 & -1 & & & & & & & \\
0 & 0 & 0 & -1 & & & & & & \\
-1 & -1 & m^2-1 & 0 & -1 & & & & & \\
0 & 0 & 0 & m^2-1 & 0 & -1 & & & & \\
\lambda_1^{n+1} & -\frac{B\varepsilon(\omega)}{\rho\nu} & C_n^{n+1}\lambda_1^{n-1}\lambda_2^2 & C_{n-1}^{n+1}\lambda_1^{n-2}\lambda_2^3 & \longrightarrow & C_n^{n+1}\lambda_1^n\lambda_2 & C_{n-1}^{n+1}\lambda_1^{n-1}\lambda_2^2 & C_{n-2}^{n+1}\lambda_1^{n-2}\lambda_2^3 & \longrightarrow & 0 \\
\Delta\varepsilon(\omega) & & C_1^{n+1}\lambda_1^n\lambda_2 & C_2^{n+1}\lambda_1^{n-1}\lambda_2^2 & \longrightarrow & C_{n-1}^{n+1}\lambda_1^2\lambda_2^{n-1} & C_n^{n+1}\lambda_1\lambda_2^n & & & \lambda_2^{n+1}
\end{vmatrix}
$$

The final determinant obtained from Δ in this manner will have in its lower left hand corner the following second order determinant

$$(22)\quad \begin{vmatrix} (h_1 + gh_3 + g^2h_5 + \cdots + g^{(n+1)/2}h_{n+2}) & (h_2 + gh_4 + g^2h_6 + \cdots + g^{(n-1)/2}h_{n+1}) \\ (gh_2 + g^2h_4 + g^3h_6 + \cdots + g^{(n+1)/2}h_{n+1}) & (h_1 + gh_3 + g^2h_5 + \cdots + g^{(n+1)/2}h_{n+2}) \end{vmatrix}.$$

The elements of the first two columns of the modified determinant Δ, other than those which appear in this second order determinant, are equal to zero. Hence the determinant Δ, for n odd, is immediately seen by Laplace's rule to be equal to the determinant (22).

For n even (including $n = 0$) we proceed in a corresponding manner and obtain in place of the determinant (22) the following determinant

$$(23)\quad \begin{vmatrix} (h_1 + gh_3 + g^2h_5 + \cdots + g^{n/2}h_{n+1}) & (h_2 + gh_4 + g^2h_6 + \cdots + g^{n/2}h_{n+2}) \\ (gh_2 + g^2h_4 + \cdots + g^{(n+2)/2}h_{n+2}) & (h_1 + gh_3 + g^2h_5 + \cdots + g^{n/2}h_{n+1}) \end{vmatrix}$$

which is now equal to the *negative* of the determinant Δ.

Expanding (22) we have

$$\Delta = (h_1 + gh_3 + \cdots + g^{(n+1)/2}h_{n+2})^2 - g(h_2 + gh_4 + \cdots + g^{(n-1)/2}h_{n+1})^2.$$

Now put $g = m^2 - 1 = e^2$. Introduce the quantity e within the second parenthesis in the above equation for Δ so that Δ is given as the difference of the squares of two expressions and hence can be written as the product of the sum and difference of these expressions. When, furthermore, we substitute for the h's their values in terms of the binomial coefficients C_r^{n+1} we obtain

$$\Delta = [\lambda_1^{n+1} + C_1^{n+1}\lambda_1^n(e\lambda_2) + C_2^{n+1}\lambda_1^{n-1}(e\lambda_2)^2 + \cdots + C_n^{n+1}\lambda_1(e\lambda_2)^n + (e\lambda_2)^{n+1}]$$
$$\times [\lambda_1^{n+1} - C_1^{n+1}\lambda_1^n(e\lambda_2) + C_2^{n+1}\lambda_1^{n-1}(e\lambda_2)^2 + \cdots - C_n^{n+1}\lambda_1(e\lambda_2)^n + (e\lambda_2)^{n+1}].$$

But the first bracket expression in this product is $(\lambda_1 + e\lambda_2)^{n+1}$ and the second is $(\lambda_1 - e\lambda_2)^{n+1}$. Hence

$$\Delta = (\lambda_1 + e\lambda_2)^{n+1}(\lambda_1 - e\lambda_2)^{n+1} = [(\lambda_1 + e\lambda_2)(\lambda_1 - e\lambda_2)]^{n+1}$$
$$= (\lambda_1^2 - e^2\lambda_2^2)^{n+1} = [\lambda_1^2 - (m^2 - 1)\lambda_2^2]^{n+1},$$

or, since n is odd, we have

$$(24)\qquad \Delta = (m^2\lambda_2^2 - 1)^{n+1}.$$

For n even we expand the determinant (23) and treat in a similar manner. Then taking account of the fact that Δ is the negative of the determinant (23) we again find that Δ is given by (24). *Hence the determinant Δ is given by* (24) *regardless of whether n is even or odd.* It is important to remember that the quantity λ_2 in (24) is taken relative to the rotated coordinate system (§3). Hence $\lambda_2 = -\sin(\alpha - \omega)$ in (24).

6. Proof that Δ does not vanish in the α interval. Consider the ratio u_n^2/c^2. We have

$$\frac{u_n^2}{c^2} = \frac{u_n^2\rho}{\gamma p} = 1 + \left(\frac{u_n^2\rho}{\gamma p} - 1\right) = 1 + \frac{\rho u_n^2 - \gamma p}{\gamma p}$$

$$= 1 - \frac{(\gamma+1)}{2\gamma p}\left[\frac{-2(\rho u_n^2 - \gamma p)}{\gamma+1}\right].$$

But from the following *remark* we can substitute

$$[p] = \frac{-2(\rho u_n^2 - \gamma p)}{\gamma+1}.$$

Hence,

$$\frac{u_n^2}{c^2} = 1 - \frac{(\gamma+1)[p]}{2\gamma p} < 1, \tag{25}$$

since $[p] > 0$ for all values of α in the interval $\alpha_0(M) < \alpha \leqq 90°$*. Also

$$\frac{u_n^2}{c^2} = \left(\frac{v^2}{c^2}\right)\frac{u_n^2}{v^2} = \frac{m^2 u_n^2}{v^2} = m^2 \sin^2\theta, \tag{26}$$

where $\theta = \alpha - \omega$. But $\sin^2\theta = \lambda_2^2$ and hence from (25) and (26) we see that $m^2\lambda_2^2 < 1$. *Hence the determinant* Δ, *given by* (24), *cannot vanish for values of* α *in the interval* $\alpha_0(M) < \alpha \leqq 90°$. *In fact we see that for* $\alpha_0(M) < \alpha \leqq 90°$ *we have*

$$\Delta > 0, \text{ for } n \text{ odd}, \qquad \Delta < 0, \text{ for } n \text{ even}.$$

Now let $\alpha \to \alpha_0(M)$ from above. What happens to the determinant Δ? To answer this question we make use of the formula[2]

$$\frac{u_n^2}{c^2} = \frac{M^2\cos^2\alpha\tan^2(\alpha-\omega)(\rho/\rho_1)}{p/p_1}. \tag{27}$$

From the formulas for the ratios ρ/ρ_1 and p/p_1 in §2 we see that $\rho/\rho_1 \to 1$ and $p/p_1 \to 1$ since $\sin\alpha \to 1/M$ as $\alpha \to \alpha_0(M)$. Also $\omega \to 0$ as $\alpha \to \alpha_0(M)$. Hence from (27) it follows that $u_n^2/c^2 \to 1$ as $\alpha \to \alpha_0(M)$. Hence $m^2\sin^2\theta$ or $m^2\lambda_2^2 \to 1$ on account of (26). But we saw above that $m^2\lambda_2^2 < 1$ in the interval $\alpha_0(M) < \alpha \leqq 90°$. Hence, using (24), it follows that *when* $\alpha \to \alpha_0(M)$ *we have*

$$\Delta \to 0 \textit{ from above for } n \textit{ odd},$$

$$\Delta \to 0 \textit{ from below for } n \textit{ even}.$$

Remark. It is sometimes convenient, as in the above instance, to express the discontinuities $[u_n]$, $[p]$ and $[\rho]$ in terms of quantities immediately behind the shock instead of in terms of quantities relative to the incident flow. For possible

* The condition $[p] > 0$ is equivalent to the condition $\rho/\rho_1 > 1$ (condition for compression shock) which was used in the derivation of the interval for the shock angle α (*loc. cit.* 2).

future reference we derive the general form of these relations (plane or three dimensional case). We begin with the relations[4]

$$[u_\alpha] = \frac{2\{\rho_1(G - u_{1n})^2 - \gamma p_1\}\nu_\alpha}{(\gamma + 1)\rho_1(G - u_{1n})},$$

$$[p] = \frac{2\{\rho_1(G - u_{1n})^2 - \gamma p_1\}}{\gamma + 1},$$

$$[\rho] = \frac{2\rho_1\{\rho_1(G - u_{1n})^2 - \gamma p_1\}}{2\gamma p_1 + (\gamma - 1)\rho_1(G - u_{1n})^2},$$

where G denotes the normal velocity of propagation of the shock wave along its normal ν directed into the region behind the shock; the other quantities in these equations have the same significance as in the preceding discussion. Let us think of the above discontinuities as being written

$$[u_\alpha] = u_{2\alpha} - u_{1\alpha}\,; \qquad [p] = p_2 - p_1\,; \qquad [\rho] = \rho_2 - \rho_1$$

so that the subscript 2 refers to quantities behind the shock line. Then we can evidently interchange the subscripts 1 and 2 in the above shock conditions provided we consider the normal ν to be directed from region 2 (region behind the shock) into region 1 (region before the shock) and hence measure the wave velocity G and normal components of the velocities u_1 and u_2 along this normal direction. If we do this and then change back to the original normal ν, directed from region 1 into region 2, which involves a change of algebraic sign in G, the components ν_α and the velocity components u_{1n} and u_{2n} we have

$$[u_\alpha] = \frac{-2\{\rho(G - u_n)^2 - \gamma p\}\nu_\alpha}{(\gamma + 1)\rho(G - u_n)},$$

$$[p] = \frac{-2\{\rho(G - u_n)^2 - \gamma p\}}{\gamma + 1},$$

$$[\rho] = \frac{-2\rho\{\rho(G - u_n)^2 - \gamma p\}}{2\gamma p + (\gamma - 1)\rho(G - u_n)^2},$$

where we have omitted the subscript 2 in conformity with above procedure. In these relations the bracket expressions $[u_\alpha]$, $[p]$ and $[\rho]$ have their previous meaning, the normal ν is directed into the region behind the shock wave and the G and u_n are measured along this normal.

In the case of a stationary wave front for which $G = 0$ the above relations become

$$[u_\alpha] = \frac{2(\rho u_n^2 - \gamma p)\nu_\alpha}{(\gamma + 1)\rho u_n},$$

$$[p] = \frac{-2(\rho u_n^2 - \gamma p)}{\gamma + 1},$$

$$[\rho] = \frac{-2\rho(\rho u_n^2 - \gamma p)}{2\gamma p + (\gamma - 1)\rho u_n^2}.$$

7. The D determinant. Suppose first that n is odd. We now carry out the first part of the process used in the evaluation of Δ in §5, i.e. we multiply the next to last column of D by $g = m^2 - 1$, add to the column two places in advance, etc. The process terminates with an addition to the first column of D. This suffices in the present case for the evaluation of D due to the presence of the zero in the second column and second row and we now immediately see that

$$D = \begin{vmatrix} (h_1 + gh_3 + g^2h_5 + \cdots + g^{(n+1)/2}h_{n+2}) - \dfrac{B\kappa_{(n)}}{\rho v} \\ (gh_2 + g^2h_4 + \cdots + g^{(n+1)/2}h_{n+1}) \qquad A_2\kappa_{(n)} \end{vmatrix}, \quad (n \text{ odd}). \tag{28}$$

When n is an even integer (including zero) we begin our process with the last column of D to obtain an addition to the first column as the final step. In this case the result is

$$D = -\begin{vmatrix} (h_1 + gh_3 + g^2h_5 + \cdots + g^{n/2}h_{n+1}) - \dfrac{B\kappa_{(n)}}{\rho v} \\ (gh_2 + g^2h_4 + \cdots + g^{(n+2)/2}h_{n+2}) \qquad A_2\kappa_{(n)} \end{vmatrix}, \quad (n \text{ even}). \tag{29}$$

To complete the evaluation of the determinants appearing in (28) and (29) we have recourse to the explicit formulas for the quantities B and A_2 (§4). Noting the following relations

$$\lambda_1 = \nu_2 = -\cos(\alpha - \omega); \qquad \lambda_2 = -\nu_1 = -\sin(\alpha - \omega)$$

$$u_t = u_n\lambda_n = u_1\lambda_1 = v\lambda_1; \qquad u_n = u_n\nu_n = u_1\nu_1 = v\nu_1$$

for purposes of easy reference we have

$$B = -\frac{4\rho_1 u_{1n} u_t}{\gamma + 1} = \frac{-4\rho u_n u_t}{\gamma + 1} = \frac{4\rho v^2 \lambda_1 \lambda_2}{\gamma + 1}.$$

Hence,

$$\frac{B}{\rho v} = \frac{4v\lambda_1\lambda_2}{\gamma + 1}. \tag{30}$$

Also,

$$\begin{aligned} A_2 &= -\frac{B\nu_2}{\rho u_n} + \frac{[\rho]}{\rho}(u_{1n}\lambda_2 - u_t\nu_2) \\ &= -\frac{B\lambda_1}{\rho u_n} + \frac{[\rho]}{\rho}\left(\frac{\rho_1 u_{1n}}{\rho_1}\lambda_2 - u_t\lambda_1\right) \\ &= \left(\frac{4\lambda_1^2}{\gamma + 1} - \frac{[\rho]}{\rho}\lambda_1^2 - \frac{[\rho]}{\rho_1}\lambda_2^2\right)v. \end{aligned} \tag{31}$$

Now expand the determinants in (28) and (29), substitute the expressions containing the binomal coefficients C_r^{n+1} for the h's and also put for $B/\rho v$ the

expression given by (30). We then obtain

$$D = \frac{4v\lambda_1\lambda_2\kappa_{(n)}}{\gamma+1}\,(gC_1^{n+1}\lambda_1^n\lambda_2 + g^2C_2^{n+1}\lambda_1^{n-2}\lambda_2^3 + \cdots + g^{(n+1)/2}C_n^{n+1}\lambda_1\lambda_2^n) + A_2\kappa_{(n)}(\lambda_1^{n+1} + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 + \cdots + g^{(n+1)/2}\lambda_2^{n+1}),\ (n \text{ odd}), \tag{32}$$

and,

$$D = -\frac{4v\lambda_1\lambda_2\kappa_{(n)}}{\gamma+1}\,(gC_1^{n+1}\lambda_1^n\lambda_2 + g^2C_3^{n+1}\lambda_1^{n-2}\lambda_2^3 + \cdots + g^{(n+2)/2}\lambda_2^{n+1}) - A_2\kappa_{(n)}(\lambda_1^{n+1} + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 + \cdots + g^{n/2}C_n^{n+1}\lambda_1\lambda_2^n),\ (n \text{ even}), \tag{33}$$

where the quantity A_2 is given by (31). In spite of the fact that these expressions are somewhat lengthy the above formulas (32) and (33) will have immediate application in §9.

We can obtain a simplified expression for the determinant D which is applicable both for n even and odd as follows. Introduce the quantity e defined by $g = m^2 - 1 = e^2$. Observe that

$$(\lambda_1 + e\lambda_2)^{n+1} = \lambda_1^{n+1} + C_1^{n+1}\lambda_1^n(e\lambda_2) + \cdots + (e\lambda_2)^{n+1},$$
$$(\lambda_1 - e\lambda_2)^{n+1} = \lambda_1^{n+1} - C_1^{n+1}\lambda_1^n(e\lambda_2) + \cdots \pm (e\lambda_2)^{n+1},$$

where the $+$ sign applies in the last term if n is odd and the $-$ sign if n is even (or zero). Then replacing g by e^2 in the expressions (32) and (33) for D we obtain

$$D = A_2\kappa_{(n)}\left[\frac{(\lambda_1 + e\lambda_2)^{n+1} + (\lambda_1 - e\lambda_2)^{n+1}}{2}\right](-1)^{n+1} + \frac{4v\lambda_1(e\lambda_2)\kappa_{(n)}}{\gamma+1}\left[\frac{(\lambda_1 + e\lambda_2)^{n+1} - (\lambda_1 - e\lambda_2)^{n+1}}{2}\right](-1)^{n+1}. \tag{34}$$

The expression (34) *for D is seen to hold in all cases, i.e. when n is zero or an even or odd integer.*

8. A lemma. In this section we prove the following

Lemma. For $m \geqq 1$ *the expression*

$$a = \left(\frac{4}{\gamma+1} - \frac{[\rho]}{\rho}\right)\lambda_1^2 - \frac{[\rho]}{\rho_1}\lambda_2^2 > 0. \tag{35}$$

We first show that the coefficient of λ_1^2 in (35) is positive. We have[1]

$$\frac{[\rho]}{\rho} = \frac{2(u_{1_n}^2 - c_1^2)}{(\gamma+1)u_{1_n}^2}.$$

Hence,

$$\frac{4}{\gamma+1} - \frac{[\rho]}{\rho} = \frac{4}{\gamma+1} - \frac{2(u_{1_n}^2 - c_1^2)}{(\gamma+1)u_{1_n}^2} = \frac{2(u_{1_n}^2 + c_1^2)}{(\gamma+1)u_{1_n}^2} > 0. \tag{36}$$

. . ow multiply the two members of (35) by v^2 to obtain

$$av^2 = \left(\frac{4}{\gamma+1} - \frac{[\rho]}{\rho}\right) u_t^2 - \frac{[\rho]}{\rho_1} u_n^2. \tag{37}$$

But,

$$[p] = \frac{\rho_1 u_{1n}^2 [\rho]}{\rho} = \frac{\rho u_n u_{1n} [\rho]}{\rho} = u_n u_{1n} [\rho] = \frac{\rho u_n^2 [\rho]}{\rho_1}. \tag{38}$$

and hence,

$$\frac{[\rho] u_n^2}{\rho_1} = \frac{[p]}{\rho} = \frac{[p]}{\gamma p} \cdot \frac{\gamma p}{\rho} = \frac{[p] c^2}{\gamma p}. \tag{39}$$

From the hypothesis $m \geqq 1$ we have $v^2 = u_t^2 + u_n^2 \geqq c^2$ so that $u_t^2 \geqq c^2 - u_n^2$. Since the coefficient of u_t^2 in the right member of (37) has been shown to be positive we will not increase the value of this member if we replace u_t^2 by $c^2 - u_n^2$. Making this substitution and also the substitution (39) we see that

$$av^2 \geqq \left(\frac{4}{\gamma+1} - \frac{[\rho]}{\rho}\right)(c^2 - u_n^2) - \frac{[p] c^2}{\gamma p}. \tag{40}$$

Then, multiplying both members of (40) by $[\rho]/\rho_1 c^2$ and again using (39), we have

$$\frac{av^2 [\rho]}{\rho_1 c^2} \geqq \left(\frac{4}{\gamma+1} - \frac{[\rho]}{\rho}\right)\left(\frac{[\rho]}{\rho_1} - \frac{[p]}{\gamma p}\right) - \frac{[p]}{\gamma p} \frac{[\rho]}{\rho_1}. \tag{41}$$

The usual form of the Rankine-Hugoniot shock condition is

$$\frac{p}{p_1} = \frac{(\gamma+1)\rho - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho}, \tag{42}$$

and from this we easily deduce the following equivalent relation

$$\frac{[p]}{\gamma p} = \frac{2[\rho]}{(\gamma+1)\rho - (\gamma-1)\rho_1}. \tag{43}$$

Now the denominator in the right member of (43) is positive since the other quantities in this equation are positive; likewise, and for the same reason, the denominator in the right member of (42) is positive. Using these facts and making the substitution (43) into the right member of (41) we easily deduce that

$$\frac{av^2}{c_2} \cdot \frac{(\gamma+1)\rho - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho} \geqq \frac{[\rho]}{\rho}.$$

Hence, using (42), we obtain

$$\frac{av^2 p}{p_1} \geqq \frac{c^2 [p]}{\rho}. \tag{44}$$

It follows now that a is positive since the other quantities in the inequality (44) are positive. This completes the proof of the above lemma.

Now $A_2 = av$. If $v = 0$ it follows that $u_n = 0$. But if $u_n = 0$ we have $[p] = 0$ from (38) which contradicts the assumption that we are dealing with a shock wave[4]. Hence $v > 0$. *It follows therefore from the above lemma that* $A_2 > 0$ *when* $m \geqq 1$; *from the results of* §2 *the inequality* $A_2 > 0$ *holds within the interval* $\alpha_0(M) < \alpha \leqq \beta(M)$.

9. The case $m \geqq 1$. For $m \geqq 1$ we have $g = m^2 - 1 \geqq 0$. Now $\lambda_1 < 0$ and $\lambda_2 < 0$. Using these inequalities and also the inequality $A_2 > 0$ it follows, from the statement at the end of §8, that the coefficients of $\kappa_{(n)}$ in the right members of (32) and (33) are positive in the interval $\alpha_0(M) < \alpha \leqq \beta(M)$. Denoting the coefficient of $\kappa_{(n)}$ by $E_n v$ we therefore have from (7), (21) and the fact that for $\alpha_0(M) < \alpha \leqq 90°$ the determinant Δ does not vanish (§6) that

$$G_n(M, \alpha) = \frac{E_n}{\Delta}, \text{ for } \alpha_0(M) < \alpha \leqq \beta(M), \tag{45}$$

where E_n is positive for all n. But $\Delta > 0$ for n odd and $\Delta < 0$ for n even (including $n = 0$) by the results of §6. *Hence in the interval* $\alpha_0(M) < \alpha \leqq \beta(M)$ *we have*

$$G_n(M, \alpha) > 0, \textit{ for } n \textit{ odd},$$

$$G_n(M, \alpha) < 0, \textit{ for } n \textit{ even}.$$

Hence there exists no singular shock direction for which the shock angle α *lies in the interval* $\alpha_0(M) < \alpha \leqq \beta(M)$.

It is easily seen that E_n is positive at $\alpha = \alpha_0(M)$. Using this fact and taking account of the behavior of the determinant Δ, as shown in §6, it follows that

$$G_n(M, \alpha) \to +\infty, \text{ as } \alpha \to \alpha_0(M), \text{ for } n \text{ odd},$$

$$G_n(M, \alpha) \to -\infty, \text{ as } \alpha \to \alpha_0(M), \text{ for } n \text{ even}.$$

10. Formula for $G_n(M, \alpha)$ when $m < 1$. For $m < 1$ we have $g = m^2 - 1 = e^2$ where $e = i\bar{e}$ and $\bar{e} = \sqrt{1 - m^2}$. Now consider the following expansions

$$\begin{aligned}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = {} & \lambda_1^{n+1} + iC_1^{n+1}\lambda_1^n(\bar{e}\lambda_2) + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 - iC_3^{n+1}\lambda_1^{n-2}(\bar{e}\lambda_2)^3 \\ & + g^2C_4^{n+1}\lambda_1^{n-3}\lambda_2^4 + iC_5^{n+1}\lambda_1^{n-4}(\bar{e}\lambda_2)^5 + \cdots \pm iC_n^{n+1}\lambda_1(\bar{e}\lambda_2)^n \\ & + g^{(n+1)/2}\lambda_2^{n+1}, \qquad (n \text{ odd}),\end{aligned}$$

and,

$$\begin{aligned}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = {} & \lambda_1^{n+1} + iC_1^{n+1}\lambda_1^n(\bar{e}\lambda_2) + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 \\ & + \cdots + g^{n/2}C_n^{n+1}\lambda_1\lambda_2^n \pm i(\bar{e}\lambda_2)^{n+1}, \qquad (n \text{ even}).\end{aligned}$$

Denoting the real part of a complex quantity by $\mathcal{R}(\,)$ and its imaginary part by $\mathcal{I}(\,)$ we see, *for* n *odd*, that

$$\mathcal{R}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = \lambda_1^{n+1} + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 + \cdots + g^{(n+1)/2}\lambda_2^{n+1},$$

$$-\bar{e}\mathcal{I}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = gC_1^{n+1}\lambda_1^n\lambda_2 + g^2C_3^{n+1}\lambda_1^{n-2}\lambda_2^3 + \cdots + g^{(n+1)/2}C_n^{n+1}\lambda_1\lambda_2^n,$$

and *for n even* we have

$$\mathcal{R}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = \lambda_1^{n+1} + gC_2^{n+1}\lambda_1^{n-1}\lambda_2^2 + \cdots + g^{n/2}C_n^{n+1}\lambda_1\lambda_2^n,$$

$$-\bar{e}\mathcal{J}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = gC_1^{n+1}\lambda_1^n\lambda_2 + g^3C_3^{n+1}\lambda_1^{n-2}\lambda_2^3 + \cdots + g^{(n+2)/2}\lambda_2^{n+1}.$$

Using these relations, formulas (32) and (33) become

$$D = \left[A_2\mathcal{R}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} - \frac{4v\lambda_1\lambda_2\bar{e}}{\gamma+1}\mathcal{J}(\lambda_1 + i\bar{e}\lambda_2)^{n+1}\right]\kappa_{(n)}, \ (n \text{ odd}), \tag{46}$$

$$D = -\left[A_2\mathcal{R}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} - \frac{4v\lambda_1\lambda_2\bar{e}}{\gamma+1}\mathcal{J}(\lambda_1 + i\bar{e}\lambda_2)^{n+1}\right]\kappa_{(n)}, \ (n \text{ even}). \tag{47}$$

Now $\lambda_2 = \lambda_1 \tan\theta$ where $\theta = \alpha - \omega$. Define $\bar{\theta}$ to be the first quadrant angle such that $\tan\bar{\theta} = \bar{e}\tan\theta$. We then have

$$\lambda_1 + i\bar{e}\lambda_2 = \lambda_1(1 + i\bar{e}\tan\theta) = \lambda_1(1 + i\tan\bar{\theta}) = \frac{\lambda_1}{\cos\theta}(\cos\bar{\theta} + i\sin\bar{\theta}).$$

Hence,

$$(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = \frac{\lambda_1^{n+1}}{\cos^{n+1}\bar{\theta}}[\cos(n+1)\bar{\theta} + i\sin(n+1)\bar{\theta}]$$

be De Moivre's theorem, and hence

$$\mathcal{R}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = \frac{\lambda_1^{n+1}\cos(n+1)\bar{\theta}}{\cos^{n+1}\bar{\theta}}, \tag{48}$$

$$\mathcal{J}(\lambda_1 + i\bar{e}\lambda_2)^{n+1} = \frac{\lambda_1^{n+1}\sin(n+1)\bar{\theta}}{\cos^{n+1}\bar{\theta}}. \tag{49}$$

Equations (48) and (49) hold for even and odd values of n. Substituting (48) and (49) into the right members of (46) and (47) we obtain

$$D = \left[\frac{A_2\lambda_1^{n+1}\cos(n+1)\bar{\theta}}{\cos^{n+1}\bar{\theta}} - \frac{4v\lambda_1^{n+2}\lambda_2\bar{e}\sin(n+1)\bar{\theta}}{(\gamma+1)\cos^{n+1}\theta}\right]\kappa_{(n)}, \ (n \text{ odd}), \tag{50}$$

$$D = -\left[\frac{A_2\lambda_1^{n+1}\cos(n+1)\bar{\theta}}{\cos^{n+1}\bar{\theta}} - \frac{4v\lambda_1^{n+2}\lambda_2\bar{e}\sin(n+1)\bar{\theta}}{(\gamma+1)\cos^{n+1}\bar{\theta}}\right]\kappa_{(n)}, \ (n \text{ even}). \tag{51}$$

Using (7), (21) and (24), making the substitutions $A_2 = av$, $\bar{e} = \sqrt{1-m^2}$ and replacing λ_1 and λ_2 by their values in terms of θ we find that both (50) and (51) give the following relation

$$G_n(M,\alpha) = \frac{\cos^{n+1}\theta}{\cos^{n+1}\bar{\theta}}\left[\frac{a\cos(n+1)\bar{\theta}}{(m^2\sin\theta - 1)^{n+1}} - \frac{2\sqrt{1-m^2}\sin 2\theta\sin(n+1)\bar{\theta}}{(\gamma+1)(m^2\sin^2\theta - 1)^{n+1}}\right], \tag{52}$$

where a is given by (35).

The relation (52) can be simplified slightly as follows. We have

$$\frac{\cos\theta}{\cos\bar{\theta}} = \left(\frac{1+\tan^2\bar{\theta}}{1+\tan^2\theta}\right)^{1/2} = \left(\frac{1+\bar{e}^2\tan^2\theta}{1+\tan^2\theta}\right)^{1/2}$$

and when the value $\bar{e}^2 = 1 - m^2$ is substituted this equation is seen to reduce to

$$\frac{\cos\theta}{\cos\bar{\theta}} = (1 - m^2\sin^2\theta)^{1/2}.$$

Making this substitution in (52) we find

$$(53)\quad G_n(M,\alpha) = \frac{(-1)^{n+1}}{(1-m^2\sin^2\theta)^{(n+1)/2}}\left[a\cos(n+1)\bar{\theta} - \frac{2\sqrt{1-m^2}}{\gamma+1}\sin 2\theta\sin(n+1)\bar{\theta}\right]$$

Equation (53) *is a general formula for* $G_n(M, \alpha)$ *when* $m < 1$, i.e. *for values of* α *in the interval* $\beta(M) < \alpha \leqq 90°$ (see §2).

11. The zeros of the functions $G_n(M, \alpha)$. From the results in §9 the zeros of the functions $G_n(M, \alpha)$ can occur only in the interval $\beta(M) < \alpha \leqq 90°$ corresponding to $m < 1$ (§2). For this α interval the formula (53) applies and we see from this formula that $G_n(M, \alpha)$ can vanish if, and only if, the bracket expression in the right member is equal to zero. Hence putting

$$b = -\frac{2\sqrt{1-m^2}}{\gamma+1}\sin 2\theta,$$

the condition for $G_n(M, \alpha)$ to vanish becomes

$$(54)\qquad a\cos(n+1)\bar{\theta} + b\sin(n+1)\bar{\theta} = 0,$$

where a and b are functions of α and M.

Since $b < 0$ the equations

$$\cos\psi = \frac{-b}{\sqrt{a^2+b^2}},\quad \sin\psi = \frac{-a}{\sqrt{a^2+b^2}}$$

define an angle ψ which is either zero, or lies within the first or fourth quadrants, and which depends on α and M. Then replacing a and b in (54) by $\sin\psi$ and $\cos\psi$ respectively the condition that $G_n(M, \alpha) = 0$ assumes the form

$$(55)\qquad \sin[(n+1)\bar{\theta} + \psi] = 0.$$

Now consider $M(> 1)$ to be fixed so that $\bar{\theta} = \bar{\theta}(\alpha)$ and $\psi = \psi(\alpha)$. Likewise the function $\bar{\varepsilon}$, which depends on M and α, becomes, for M fixed, a function $\bar{\varepsilon} = \bar{\varepsilon}(\alpha)$. From the algebraic character of the shock conditions it is clear that the functions $\bar{\theta}(\alpha)$, $\psi(\alpha)$ and $\bar{\varepsilon}(\alpha)$ are differentiable functions of α; moreover it is evident that the right member of the equation defining $\bar{\theta}$, namely

$$(56)\qquad \tan\bar{\theta} = \bar{\varepsilon}\tan\theta = \sqrt{1-m^2}\tan(\alpha-\omega)$$

can have but a finite number of turning points in the interval $\beta(M) < \alpha \leqq 90°$ and hence that *the derivative* $d\bar{\theta}/d\alpha$ *can vanish* (*or become infinite*) *at most for a finite number of values of* α *in this interval.* These general observations, by which

explicit calculations are avoided, will suffice for our discussion of the distribution of the zeros of the functions $G_n(M, \alpha)$.

If $\alpha = 90°$, then $\omega = 0$ (*loc. cit.* 2) and hence from (56) $\bar{\theta} = 90°$. But (54) is satisfied by $n = 0$ and $\bar{\theta} = 90°$; hence $G_0(M, 90°) = 0$. Since $\alpha = 90°$ gives a zero of the function $G_0(M, \alpha)$, for any value of M, this value of α can be disregarded in our consideration of the distribution of the zeros of the functions $G_n(M, \alpha)$ and our attention confined to the open interval $\beta(M) < \alpha < 90°$.

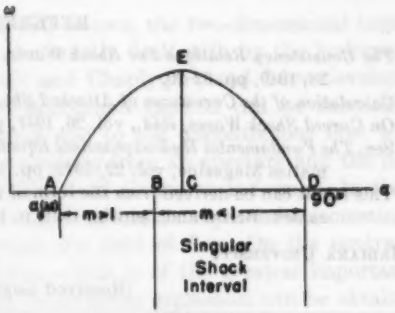

FIG. 4

Now let α_1 be any value of α in the interval $\beta(M) < \alpha < 90°$. Within this interval consider an interval containing α_1 defined by $|\alpha - \alpha_1| < \delta$. Take α_2 to be a value of α in this δ interval such that $d\bar{\theta}/d\alpha \neq 0$ at α_2. Select an interval $|\alpha - \alpha_2| < \epsilon$, lying within the δ interval, and within which $d\bar{\theta}/d\alpha \neq 0$. Since $\bar{\theta}(\alpha)$ is a monotonically increasing or decreasing function of α in the ϵ interval we can find a value α_3 in this latter interval such that $\bar{\theta}(\alpha_3)$ is *irrational*.

Now consider the set of values

$$\Omega(n, \alpha_3) = (n + 1)\bar{\theta}(\alpha_3) + \psi(\alpha_3), \qquad n = 0, 1, 2, 3, \cdots. \tag{57}$$

Since $\bar{\theta}(\alpha_3)$ is irrational the Ω's can be reduced, mod. 360, to a set of values $\bar{\Omega}(n)$ dense[5] in the interval $-180 < \bar{\Omega} \leqq 180$. Hence we can select an increasing sequence of integers n_k such that $\bar{\Omega}(n_k) > 0$ and $\bar{\Omega}(n_k) \to 0$ as $k \to \infty$; then the $\Omega(n_k, \alpha_3)$ will appear as first quadrant angles so that $\sin \Omega(n_k, \alpha_3) > 0$ and $\sin \Omega(n_k, \alpha_3) \to 0$ as $k \to \infty$. By taking k sufficiently large it is evident, since $\bar{\theta}(\alpha)$ increases or decreases continuously as a function of α within the above ϵ interval, that a small change in α from the value α_3 to a value α_4 lying in the ϵ interval can be made such that $\Omega(n_k, \alpha_4)$ will be an angle within the fourth quadrant. Hence for this value of k we will have

$$\sin \Omega(n_k, \alpha_3) > 0, \qquad \sin \Omega(n_k, \alpha_4) < 0.$$

Hence there must be some value of $\alpha = \alpha_5$ between α_3 and α_4 and lying in the ϵ interval such that $\sin \Omega(n_k, \alpha_5) = 0$. *We have therefore shown that if $\alpha = \alpha_1$ is a value of α in the interval $\beta(M) < \alpha < 90°$ any neighborhood of α_1 contains a value of α at which* (55) *is satisfied for some integer n. In other words the zeros of the functions $G_n(M, \alpha)$,* i.e. *the singular shock angles α, are dense in the interval $\beta(M) < \alpha < 90°$.*

The figure shows the nature of the graph of the angle ω for a given value of $M > 1$ and the location of the singular shock interval. Point A corresponds to the value $\alpha = \alpha_0(M)$ and B to the value $\alpha = \beta(M)$ at which $m = 1$. In the open interval AB no singular shock angles α occur, nor is there a singular shock angle

at the end point B. The singular shock angles are dense in the open interval BD and a singular angle occurs at D. The ω curve reaches its maximum height at a point E corresponding to a value of α, represented by the point C, in the singular shock interval*.

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(Received August 8, 1948)

* This conclusion is reached by observation of the graphs of the functions ω and $-G_0(M, \alpha)$ shown in the paper *Calculation of the Curvatures of Attached Shock Waves*, *loc. cit.* 2.

APPLICATION OF THE W.K.B. METHOD TO THE FLOW OF A COMPRESSIBLE FLUID, I

By Isao Imai

1. Introduction and summary. As is well known, the two-dimensional irrotational flow of a compressible fluid can be exactly dealt with by the hodograph method, due originally to Molenbroek[9] and Chaplygin[2], and later developed by Ringleb[10]. The application of the hodograph method can be successful, for example, to the flow round a corner and to the discontinuous flow with free stream-lines, such as the free jet emerging from an aperture and the flow impinging on a flat plate, behind which a dead air region is formed. In these cases of flow the stream function can be expressed as a unique series (sometimes finite, but generally infinite) throughout the field of flow. On the contrary, for the continuous flow past an obstacle, which is of the greatest importance in aerodynamical applications, no such single series expression can be obtained for the stream function. Thus the exact treatment of the flow past an obstacle at high speeds, with partly subsonic and partly supersonic region, is very difficult.

For the approximate treatment, however, there is available the well-known method of Kármán and Tsien[7, 12]. However, in view of the approximative nature of the hypothetical gas introduced, the method seems to be less accurate near the critical Mach number.

In this paper the fundamental equations of the hodograph method will be expressed in terms of new variables suggested by the W.K.B. method.* In the case of subsonic flow past an obstacle approximate treatment similar to Kármán-Tsien's method can be developed which is believed to be very accurate even near the critical Mach number. Some numerical examples will be given in a forthcoming paper by H. Hasimoto and the present author.

In the supersonic case the equations of motion in terms of new variables have the advantage of giving immediately the equations for the characteristics.

Further, the equations are also very convenient for the approximate treatment of the transonic case in which the flow is partly subsonic and partly supersonic in the region considered. Such a treatment will be given in another paper.

2. Fundamental equations. The two-dimensional irrotational flow of a non-viscous compressible fluid in the xy-plane is governed by the equations:

$$u = \partial\Phi/\partial x, \qquad v = \partial\Phi/\partial y, \tag{2.1}$$

$$u = (\rho_0/\rho)(\partial\Psi/\partial y), \qquad v = -(\rho_0/\rho)(\partial\Psi/\partial x), \tag{2.2}$$

* Quite recently the author has received from Prof. S. Bergman a series of interesting papers dealing with the compressible flow. It was found that the new variables τ, ϕ, ψ introduced in this paper are just the same as those employed by him with the notations λ, ϕ^*, ψ^*[1]. However, the present paper will not be useless, in view of the fact that the reason for the introduction of the new variables seems to present some interest, and that the subsequent treatment is believed to be quite different from that of Prof. Bergman. The author wishes to thank Prof. Bergman for favouring him with the reprints of his papers of greatest importance and interest.

where (u, v) is the velocity vector, Φ the velocity potential, Ψ the stream function, ρ the density, and ρ_0 a certain constant having the dimensions of the density, which may be conveniently taken to be the density of the fluid at rest. Denoting the magnitude of the velocity vector by q and the angle between the x-axis and the velocity vector by θ, we have

$$u = q\cos\theta, \qquad v = q\sin\theta. \tag{2.3}$$

Let us assume that the pressure, p, of the gas is a function of the density ρ only. Then Bernoulli's theorem can be expressed in the form:

$$qdq + dp/\rho = 0, \tag{2.4}$$

whence p, and therefore ρ can be expressed as a definite function of q. Thus, remembering that $q^2 = u^2 + v^2$, we may consider (2.1) and (2.2) as a system of non-linear partial differential equations for Φ and Ψ with x and y as independent variables.

Let us now go over to the hodograph plane, taking q, θ as independent variables. From (2.1) and (2.2) we have

$$d\Phi = udx + vdy = q(\cos\theta\, dx + \sin\theta\, dy),$$

$$\rho_0 d\Psi = \rho(-vdx + udy) = q\rho(-\sin\theta\, dx + \cos\theta\, dy).$$

Solving for dx and dy, we have

$$dx + idy = dz = q^{-1}e^{i\theta}[d\Phi + i(\rho_0/\rho)]d\Psi], \tag{2.5}$$

with $z = x + iy$. Hence, the equality: $\partial^2 z/\partial q\partial\theta = \partial^2 z/\partial\theta\partial q$ gives, on reduction and separating into real and imaginary parts,

$$\frac{\partial\Phi}{\partial q} = q\frac{d}{dq}\left(\frac{\rho_0}{q\rho}\right)\frac{\partial\Psi}{\partial\theta}, \qquad \frac{\partial\Phi}{\partial\theta} = q\frac{\rho_0}{\rho}\frac{\partial\Psi}{\partial q}, \tag{2.6}$$

which are linear partial differential equations for Φ and Ψ as functions of q and θ, and form the basis of the hodograph method.

(2.6) can be further simplified by introducing a new variable t defined as:

$$t = \int (\rho/\rho_0)dq/q. \tag{2.7}$$

Thus (2.6) become

$$\partial\Phi/\partial\theta = \partial\Psi/\partial t, \qquad \partial\Phi/\partial t = -K\partial\Psi/\partial\theta, \tag{2.8}$$

with

$$K = \left(1 - \frac{q^2}{c^2}\right)\left(\frac{\rho_0}{\rho}\right)^2, \tag{2.9}$$

where $c = (dp/d\rho)^{\frac{1}{2}}$ is the local velocity of sound. K is, of course, a fixed function of q and, therefore, of t only. Elimination of Φ from (2.8) gives at once

$$\partial^2\Psi/\partial t^2 + K\partial^2\Psi/\partial\theta^2 = 0. \tag{2.10}$$

As usual, particular solutions of (2.10) can be obtained in the form:

$$\Psi = T_n(t) \cos n\theta, \qquad T_n(t) \sin n\theta, \tag{2.11}$$

where n is an arbitrary constant and $T_n(t)$ is a solution of the ordinary differential equation:

$$d^2T_n/dt^2 - n^2KT_n = 0. \tag{2.12}$$

It may readily be inferred that the general solution of (2.10) can be constructed by summing up the particular solutions with different n.

3. Application of the W.K.B. method. As mentioned above, exact solutions of the differential equations for compressible flow can, in principle, be obtained in the form:

$$\Psi = \Sigma\, T_n(t)(A_n \cos n\theta + B_n \sin n\theta). \tag{3.1}$$

However, for practically important cases of flow, the series is, in general, infinite, and it becomes of basic importance to test the convergence of the series. Sometimes we are even compelled to deal with divergent series. Indeed, in the case of an obstacle submerged in a uniform flow of unlimited extent, Ψ diverges when $\theta \to 0$, $t \to t_\infty$, where t_∞ corresponds to the undisturbed flow, since Ψ increases unboundedly with the distance from the obstacle.

In order to see the convergency property of (3.1), it is essential to have information concerning the asymptotic properties of $T_n(t)$ for $n \to \infty$. This may be furnished by the W.K.B. method well-known in the quantum mechanics.

Putting

$$\tau = -\int K^{\frac{1}{2}}\, dt = -\int \mu\, dq/q, \tag{3.2}$$

and

$$T_n(t) = K^{-\frac{1}{4}} W_n(\tau), \tag{3.3}$$

where

$$\mu = (1 - q^2/c^2)^{\frac{1}{2}}, \tag{3.4}$$

(2.12) becomes

$$d^2W_n/d\tau^2 - n^2W_n = kW_n, \tag{3.5}$$

where

$$k = K^{-\frac{1}{4}} d^2K^{\frac{1}{4}}/d\tau^2 = -K^{-\frac{1}{4}} d^2K^{-\frac{1}{4}}/dt^2. \tag{3.6}$$

k is naturally a definite function of τ to be determined from the property of the gas under consideration. As $n \to \infty$, kW_n may be neglected in comparison with n^2W_n, so that W_n will be asymptotically given by

$$W_n \sim Ae^{n\tau} + Be^{-n\tau}, \tag{3.7}$$

which is the result of the W.K.B. method.

Suggested by the above result, we shall introduce new variables τ, ϕ, and ψ such that

$$\Phi = K^{\frac{1}{2}}\phi, \qquad \Psi = K^{-\frac{1}{2}}\psi. \tag{3.8}$$

Then (2.8) become

$$\partial\phi/\partial\tau - \partial\psi/\partial\theta = -K^{-\frac{1}{2}}(dK^{\frac{1}{2}}/d\tau)\phi, \tag{3.9}$$

$$\partial\phi/\partial\theta + \partial\psi/\partial\tau = -K^{\frac{1}{2}}(dK^{-\frac{1}{2}}/d\tau)\psi. \tag{3.10}$$

Eliminating ψ or ϕ gives respectively

$$\partial^2\phi/\partial\tau^2 + \partial^2\phi/\partial\theta^2 = K^{\frac{1}{2}}(d^2K^{-\frac{1}{2}}/d\tau^2)\phi, \tag{3.11}$$

$$\partial^2\psi/\partial\tau^2 + \partial^2\psi/\partial\theta^2 = K^{-\frac{1}{2}}(d^2K^{\frac{1}{2}}/d\tau^2)\psi. \tag{3.12}$$

It may be added that, in terms of the independent variables τ and θ, (2.8) can be rewritten in the form:

$$\partial\Phi/\partial\tau = K^{\frac{1}{2}}\partial\Psi/\partial\theta, \qquad \partial\Phi/\partial\theta = -K^{\frac{1}{2}}\partial\Psi/\partial\tau. \tag{3.13}$$

4. New method of approximation for flow past an obstacle. In the following we shall confine ourselves to the uniform flow past an obstacle. We now consider (3.12), which may be rewritten in the form:

$$\Delta\psi = k\psi, \qquad \Delta \equiv \partial^2/\partial\tau^2 + \partial^2/\partial\theta^2, \tag{4.1}$$

k being given by (3.6). Following facts are especially to be noted.

(i) For flow at moderate speeds K is nearly constant, and consequently k is very small.

(ii) On the surface of the obstacle we may take $\Psi = 0$, and therefore $\psi = 0$. Hence, near the obstacle $k\psi$ will be very small.

(iii) At great distances from the obstacle, ψ is very large. But, it can be shown without difficulty that $\Delta\psi$ becomes infinite of order higher than ψ. As a consequence, $k\psi$ may be safely neglected compared with $\Delta\psi$.

From these considerations, it will be natural to expect that (4.1) can be very closely approximated by

$$\Delta\psi = 0. \tag{4.2}$$

The general solution of (4.2) is given by

$$\psi = (1/2i)[f(\sigma) - \bar{f}(\bar{\sigma})], \qquad \sigma = \tau + i\theta, \tag{4.3}$$

where $f(\sigma)$ is an arbitrary analytic function of the complex variable σ.

Remembering (3.8), (3.9), and (3.10), we have

$$d\Phi = d(K^{\frac{1}{2}}\phi) = K^{\frac{1}{2}}\left\{\frac{\partial\psi}{\partial\theta}\,d\tau - \frac{\partial\psi}{\partial\tau}\,d\theta - K^{\frac{1}{2}}\,\frac{d}{d\tau}\,(K^{-\frac{1}{2}})\psi\,d\theta\right\}.$$

On the profile $\Psi = 0$, and hence (2.5) reduces to

$$dz = q^{-1}K^{\frac{1}{2}}e^{i\theta}[(\partial\psi/\partial\theta)d\tau - (\partial\psi/\partial\tau)d\theta]. \tag{4.4}$$

Substituting the approximate solution (4.3), we get, for the coordinates of the profile,

$$dz_P = q^{-1}K^{\frac{1}{4}}e^{i\theta}df. \tag{4.5}$$

We can now proceed just as in the hodograph method of Kármán and Tsien. As is well known, the analytic function $f(\sigma)$ represents an incompressible flow in a certain ζ-plane, if we put

$$df/d\zeta = Ce^{-\sigma} = Qe^{-i\theta}, \qquad Q = Ce^{-\tau}, \tag{4.6}$$

where Q and θ are respectively the magnitude and the angle of inclination of the velocity vector of the incompressible flow in the ζ-plane. It may be interesting to remark that the function Q has already been introduced in a previous paper[6] from quite a different viewpoint.

Combining (4.5) and (4.6), we get finally

$$dz_P = q^{-1}QK^{\frac{1}{4}}d\zeta_P, \tag{4.7}$$

which gives the correspondence between the profiles ($\Psi = 0$) in the z-plane and the ζ-plane.

For an arbitrary point in the field of flow, we have dz from (2.5). Neglecting $(K^{-\frac{1}{4}})'\psi$ and using the approximation $\psi = \Im f(\sigma)$, we can transform this as follows.

$$\begin{aligned} dz &= \tfrac{1}{2}q^{-1}e^{i\theta}\{K^{\frac{1}{4}}(df + d\bar{f}) + (\rho_0/\rho)K^{-\frac{1}{4}}(df - d\bar{f})\} \\ &= \frac{QK^{\frac{1}{4}}}{2q}\left(1 + \frac{1}{\mu}\right)d\zeta + \frac{K^{\frac{1}{4}}}{2qQ}\left(1 - \frac{1}{\mu}\right)\left(\frac{d\bar{f}}{d\bar{\zeta}}\right)^2 d\bar{\zeta}, \end{aligned} \tag{4.8}$$

where the relation $K = \mu^2(\rho_0/\rho)^2$ and $df/d\zeta = Qe^{-i\theta}$ have been employed.

5. Calculation of velocity and pressure over an obstacle in a uniform flow. We take an arbitrary profile P_0 in the ζ-plane and consider a uniform flow of an incompressible fluid past it. The velocity Q on the profile P_0 can be easily calculated by some current method, such as Theodorsen's method[11] or the method proposed by the present author[3, 5]. By the use of numerical tables q and then $(Q/q)K^{\frac{1}{4}}$ can be determined from Q for any point $\zeta = \zeta_P$ on the profile. Hence, on integrating (4.7), the form of the profile P in the physical z-plane can be calculated. The velocity q of the compressible flow along the profile P can thus be found. Calculation of the pressure from the velocity is quite an easy task by the use of a suitable numerical table.

In case the profile P_0 is not symmetrical with respect to the oncoming flow, there would usually be a circulation round P_0. Then the profile P in the physical z-plane is not closed as was already pointed out by Kármán and Tsien. In such a case we may proceed just as proposed in a previous paper[4].* Thus we consider a second ζ'-plane of incompressible flow, such that

$$d\zeta/d\zeta' = 1 + (k_1/\zeta') + (k_2/\zeta'^2) + \cdots. \tag{5.1}$$

* At the suggestion of the author, Mr. Ōyama[9] investigated the compressible flow with circulation past a profile derived from an elliptic cylinder of thickness ratio 1/10, at various angles of incidence, on the basis of the approximate method of Kármán and Tsien.

A closed curve P' in the ζ'-plane is mapped on a non-closed curve P_0 in the ζ-plane, and then on a closed curve P in the physical z-plane by assigning a suitable value to k_1. The relation between the velocity in the ζ- and the ζ'-plane is naturally given by

$$Q = |df/d\zeta| = |df/d\zeta'||d\zeta'/d\zeta| = Q'|d\zeta/d\zeta'|^{-1}. \tag{5.2}$$

6. Thin profile. It is very difficult to calculate the velocity distribution directly for an arbitrarily given profile by the above-mentioned procedure. We should rather start from a suitably chosen profile P_0 and find the velocity on the profile derived from P_0 by means of (4.7). Such a defect is well known to be inherent in the hodograph method. For a moderately thin aerofoil, however, the numerical labour is considerably reduced, since the profiles P and P_0 are then nearly similar to each other, so that the numerical integration (4.7) can be dispensed with.

(4.7) can be written in the form:

$$dz = \lambda d\zeta, \qquad \lambda(q) = q^{-1}QK^{\frac{1}{2}}. \tag{6.1}$$

On a thin profile the velocity q is nearly constant: $q \fallingdotseq q_1$ (we may take q_1 as the undisturbed flow velocity q_∞), so that $\lambda \fallingdotseq \lambda(q_1) = \lambda_1$. Thus

$$\lambda - \lambda_1 = O(\epsilon),$$

ϵ being a small quantity representing the order of magnitude of the thickness of the profile. From (6.1) we have

$$dz = \lambda_1 d\zeta + (\lambda - \lambda_1)d\zeta,$$

whence

$$z = \lambda_1\zeta + \int(\lambda - \lambda_1)d\xi + i\int(\lambda - \lambda_1)d\eta. \tag{6.2}$$

For a thin profile $\eta = O(\epsilon)$, and hence

$$\int(\lambda - \lambda_1)d\eta = O(\epsilon^2).$$

Separating real and imaginary parts of (6.2), we have

$$x = \lambda_1\xi + O(\epsilon), \qquad y = \lambda_1\eta + O(\epsilon^2). \tag{6.3}$$

Let the profile P_0 in the ζ-plane be given by

$$\eta = g(\xi), \qquad g(\xi) = O(\epsilon). \tag{6.4}$$

Then, by (6.3), we get

$$\begin{aligned} y/\lambda_1 &= \eta + O(\epsilon^2) = g(\xi) + O(\epsilon^2) = g[x/\lambda_1 + O(\epsilon)] + O(\epsilon^2) \\ &= g(x/\lambda_1) + g'(x/\lambda_1)O(\epsilon) + O(\epsilon^2). \end{aligned}$$

Since $g'(\xi)$ is $O(\epsilon)$ just as $\eta = g(\xi)$, we get finally

$$y/\lambda_1 = g(x/\lambda_1) + O(\epsilon^2). \tag{6.5}$$

Hence it will be obvious that, neglecting small quantities of order $O(\epsilon^2)$, the profile P in the z-plane and the profile P_0 in the ζ-plane are similar to each other and in the ratio λ_1: 1. Therefore the velocity q on a thin profile in a compressible flow can be directly obtained from the velocity on the same profile in an incompressible flow, by using a numerical table q vs. Q.

It will be of interest to remark that such an approximate method of calculating velocity distribution is just the one put forward by the present author from quite a different standpoint[6].

7. Adiabatic gas flow. We are, in general, concerned with the flow of gas obeying the adiabatic law:

$$p \propto \rho^\gamma, \tag{7.1}$$

where γ is the ratio of the specific heats. In this case we have

$$p/p_0 = (1 - a^2q^2)^{\gamma/(\gamma-1)}, \qquad a^2 = (\gamma - 1)/(\gamma + 1) \tag{7.2}$$

$$\rho/\rho_0 = (1 - a^2q^2)^{1/(\gamma-1)}, \tag{7.3}$$

$$c^2 = \tfrac{1}{2}(\gamma + 1) - \tfrac{1}{2}(\gamma - 1)q^2 = \tfrac{1}{2}(\gamma + 1)(1 - a^2q^2), \tag{7.4}$$

where the critical velocity c^*, at which the fluid velocity is equal to the local velocity of sound, is taken as the unit of velocity, and p_0 and ρ_0 are respectively the pressure and the density at the stagnation point: $q = 0$. Also we have

$$\mu = \left(1 - \frac{q^2}{c^2}\right)^{\frac{1}{2}} = \left(\frac{1 - q^2}{1 - a^2q^2}\right)^{\frac{1}{2}}, \tag{7.5}$$

$$K = \mu^2(\rho_0/\rho)^2 = (1 - q^2)(1 - a^2q^2)^{-1/a^2}, \tag{7.6}$$

$$\begin{aligned}\tau &= -\int_1^q \mu \frac{dq}{q} = -\int_1^q \left(\frac{1 - q^2}{1 - a^2q^2}\right)^{\frac{1}{2}} \frac{dq}{q} \\ &= \tanh^{-1}\mu - a^{-1}\tanh^{-1} a\mu.\end{aligned} \tag{7.7}$$

Hence

$$\begin{aligned}Q = Ce^{-\tau} &= C\left(\frac{1 + a\mu}{1 - a\mu}\right)^{1/2a}\left(\frac{1 - \mu}{1 + \mu}\right)^{\frac{1}{2}} \\ &= Cq(1 - a^2q^2)^{(1-a)/2a}(1 + a\mu)^{1/a}(1 + \mu)^{-1},\end{aligned} \tag{7.8}$$

where C is an arbitrary constant, which can be conveniently determined by the requirement that $Q \to q$ as $q \to 0$. Thus

$$C = 2(1 + a)^{-1/a}. \tag{7.9}$$

Finally (3.6) can be written as

$$\begin{aligned}k(q) = K^{-1/4}\frac{d^2}{d\tau^2}(K^{1/4}) &= -\frac{1 - a^2}{4}q^4\frac{8 - 2(1 + 3a^2)q^2 - (1 - a^2)q^4}{(1 - q^2)^2(1 - a^2q^2)} \\ &= -\frac{4}{\gamma + 1}q^4\left\{1 + \frac{6\gamma + 5}{2(\gamma + 1)}q^2 + \frac{24\gamma^2 + 39\gamma + 19}{4(\gamma + 1)^2}q^4 + \cdots\right\}.\end{aligned} \tag{7.10}$$

It will be seen that $k(q)$ is very small if q is small. In fact, if the Mach number of the undisturbed flow is denoted by $M = q_\infty/c_\infty$, we have, from (7.4),

$$q_\infty^2 = \tfrac{1}{2}(\gamma + 1)M^2[1 + \tfrac{1}{2}(\gamma - 1)M^2]^{-1}, \tag{7.11}$$

so that

$$k(q) \sim -\frac{4}{\gamma + 1} q_\infty^4 \left(\frac{q}{q_\infty}\right)^4 \sim -(\gamma + 1)M^4 \left(\frac{q}{q_\infty}\right)^4.$$

Thus, if $O(M^4)$ can be neglected, $k(q)$ may be safely taken to be zero, as mentioned in §4, (i).

$k(q)$ has a pole at $q = 1$, and therefore the method of approximation developed in this paper will be inaccurate near the transonic region: $q \fallingdotseq 1$.

8. Supersonic flow. The above discussion is, in principle, applicable to the supersonic flow as well as the subsonic flow, but, in view of the appearance of imaginary quantities, it will be convenient to treat the supersonic case separately.

For the supersonic flow $q > c$, and taking

$$1 - (q/c) = e^{-\pi i}[(q/c) - 1], \tag{8.1}$$

we have

$$\mu = [1 - (q/c)^2]^{\frac{1}{2}} = e^{-\frac{1}{2}\pi i}\,|\,\mu\,|. \tag{8.2}$$

From (3.2) we have ($c^* = 1$)

$$\tau = e^{-\pi i}\int_1^q \mu dq/q = e^{-\frac{1}{2}\pi i}\int_1^q |\,\mu\,|\,dq/q = i^{-3}\,|\,\tau\,|. \tag{8.3}$$

From (2.9) we have

$$K = \mu^2(\rho_0/\rho)^2 = i^{-2}\,|\,\mu\,|^2(\rho_0/\rho)^2 = i^{-2}\,|\,K\,|. \tag{8.4}$$

Also (3.8) can be written

$$\Phi = K^{\frac{1}{4}}\phi = i^{-\frac{1}{2}}\,|\,K\,|^{\frac{1}{4}}\phi = |\,K\,|^{\frac{1}{4}}\,|\,\phi\,|, \tag{8.5}$$

$$\Psi = K^{-\frac{1}{4}}\psi = i^{\frac{1}{2}}\,|\,K\,|^{-\frac{1}{4}}\psi = |\,K\,|^{-\frac{1}{4}}\,|\,\psi\,|, \tag{8.6}$$

where

$$\phi = i^{\frac{1}{2}}\,|\,\phi\,|, \qquad \psi = i^{-\frac{1}{2}}\,|\,\psi\,|. \tag{8.7}$$

Substituting (8.3), (8.4), and (8.7) into (3.9) and (3.10), we get

$$\partial\phi/\partial\tau - \partial\psi/\partial\theta = -K^{-1}(dK^{\frac{1}{4}}/d\tau)\phi, \tag{8.8}$$

$$\partial\phi/\partial\theta - \partial\psi/\partial\tau = K^{\frac{1}{4}}(dK^{-\frac{1}{4}}/d\tau)\psi, \tag{8.9}$$

where the symbol $|\ |$, for the absolute value, has been dropped for simplicity. Also we have, in place of (3.11) and (3.12),

$$\partial^2\phi/\partial\tau^2 - \partial^2\phi/\partial\theta^2 = K^{\frac{1}{4}}(d^2K^{-\frac{1}{4}}/d\tau^2)\phi, \tag{8.10}$$

$$\partial^2\psi/\partial\tau^2 - \partial^2\psi/\partial\theta^2 = K^{-\frac{1}{4}}(d^2K^{\frac{1}{4}}/d\tau^2)\psi. \tag{8.11}$$

It will readily be seen that the characteristics of the simultaneous equations (8.8) and (8.9) are the same as those of

$$\partial\phi/\partial\tau - \partial\psi/\partial\theta = 0, \qquad \partial\phi/\partial\theta - \partial\psi/\partial\tau = 0. \tag{8.12}$$

Thus the characteristics are given by

$$\tau \pm \theta = \text{const.} \tag{8.13}$$

Now (3.13) is replaced by

$$\partial\Phi/\partial\tau = K^{\frac{1}{2}}\partial\Psi/\partial\theta, \qquad \partial\Phi/\partial\theta = K^{\frac{1}{2}}\partial\Psi/\partial\tau. \tag{8.14}$$

Putting

$$\xi = \tau + \theta, \qquad \eta = \tau - \theta,$$

we have, from (8.14),

$$\partial\Phi/\partial\xi = -K^{\frac{1}{2}}\partial\Psi/\partial\xi, \qquad \partial\Phi/\partial\eta = K^{\frac{1}{2}}\partial\Psi/\partial\eta. \tag{8.15}$$

Hence (2.5) can be written as:

$$\begin{aligned} dz &= q^{-1}e^{i\theta}[d\Phi + i(\rho_0/\rho)d\Psi] \\ &= \frac{1}{q}e^{i\theta}\left\{-(\mu - i)\frac{\rho_0}{\rho}\frac{\partial\Psi}{\partial\xi}\,d\xi + (\mu + i)\frac{\rho_0}{\rho}\frac{\partial\Psi}{\partial\eta}\,d\eta\right\}. \end{aligned}$$

Now we have

$$\mu \pm i = (1 + \mu^2)^{\frac{1}{2}}e^{\pm i\alpha} = (q/c)e^{\pm i\alpha},$$

where $\alpha = \cot^{-1}\mu$ is the Mach angle. Thus

$$dz = \frac{\rho_0}{\rho c}\,e^{i\theta}\left(e^{i\alpha}\frac{\partial\Psi}{\partial\eta}\,d\eta - e^{-i\alpha}\frac{\partial\Psi}{\partial\xi}\,d\xi\right). \tag{8.16}$$

The two families of characteristics $\xi = \text{const}$ and $\eta = \text{const}$ are respectively given by

$$dz = \frac{\rho_0}{\rho c}\,e^{i(\theta+\alpha)}\frac{\partial\Psi}{\partial\eta}\,d\eta, \qquad dz = -\frac{\rho_0}{\rho c}\,e^{i(\theta-\alpha)}\frac{\partial\Psi}{\partial\xi}\,d\xi,$$

so that the angle between the velocity vector and each one of the characteristics is equal to the Mach angle $\alpha = \cot^{-1}\mu$, as should be expected.

In the case of adiabatic gas flow p/p_0, ρ/ρ_0, and c^2 are given by (7.2), (7.3), and (7.4) respectively. Remembering that we have simply written μ, K, τ for the quantities $|\mu|$, $|K|$, $|\tau|$, we have

$$\mu = \left(\frac{q^2 - 1}{1 - a^2q^2}\right)^{\frac{1}{2}}, \tag{8.17}$$

$$K = (q^2 - 1)(1 - a^2q^2)^{-1/a^2}, \tag{8.18}$$

$$\tau = a^{-1}\tan^{-1}a\mu - \tan^{-1}\mu. \tag{8.19}$$

Substituting (8.19) into (8.13), we may at once recognize the well-known equation for the characteristic curves in the hodograph plane for the case of adiabatic flow.

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(Received January 17, 1949)

PROPERTIES OF THE CONFLUENT HYPERGEOMETRIC FUNCTION*

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The confluent hypergeometric functions have proved useful in many branches of physics. They have been used in such problems involving diffusion and sedimentation, as isotope separation and protein molecular weight determinations in the ultracentrifuge. The solution of the equation for the velocity distribution of electrons in high frequency gas discharges may frequently be expressed in terms of these functions. The high frequency breakdown electric field may then be predicted theoretically for gases by the use of such solutions together with kinetic theory.

This report presents some of the properties of the confluent hypergeometric functions together with six-figure tables of the functions for the parameter values $\gamma = 0.5, 1.0, 1.5$, and 2.0 and $\alpha = 0.001, 0.01, 0.05, 0.1, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9$, and 1.0. Tables of the second solution of the differential equation for $\gamma = 1, 2$, and 3 at the same values of α are also included.† The tables give values for arguments up to 8 by steps of 0.5. For values of the argument above 8, the asymptotic expansions suffice.

1. Definitions. The confluent hypergeometric function $M(\alpha; \gamma; z)$ is defined as the solution, bounded at the origin, of the second order linear homogeneous differential equation

$$z\, d^2M/dz^2 + (\gamma - z)\, dM/dz - \alpha M = 0 \tag{1}$$

where γ, α, and z are unrestricted. Equation (1) has a regular singularity at the origin and an irregular singularity at infinity.

If γ is not integral, a second solution of Eq. (1) is given by

$$W(\alpha; \gamma; z) = z^{1-\gamma} M(\alpha - \gamma + 1; 2 - \gamma; z). \tag{2}$$

If γ is integral, a second solution is given by‡

$$W(\alpha;\gamma;z) = M(\alpha;\gamma;z)\{\ln z + \psi(1-\alpha) - \psi(\gamma) + C\} + \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(\gamma)B_n z^n}{\Gamma(\alpha)\Gamma(n+\gamma)n!}$$

$$+ (-1)^{\gamma} \sum_{n=0}^{\gamma-2} \frac{\Gamma(\gamma)\Gamma(n+\alpha-\gamma+1)\Gamma(\gamma-n-1)(-1)^n}{\Gamma(\alpha)n!\, z^{\gamma-n-1}}, \tag{3}$$

where

$$\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha), \qquad C = .577216$$

* This work has been supported in part by the Signal Corps, the Air Materiel Command, and O.N.R.

† The notation used in this report is that of E. Jahnke and F. Emde, "Tables of Functions", Teubner, Leipzig, 1933.

‡ W. J. Archibald, Phil. Mag. (London) 7, *26*, 419 (1938).

TABLE I: $M(\alpha; \frac{1}{2}; z)$

z\α	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00119	1.01194	1.06009	1.12121	1.24661	1.31090	1.37626	1.51027
1.0	1.00289	1.02901	1.14709	1.29939	1.62003	1.78859	1.96279	2.32856
1.5	1.00536	1.05388	1.27525	1.56542	2.19240	2.53016	2.88475	3.64646
2.0	1.00901	1.09069	1.46683	1.96791	3.07856	3.69109	4.34381	5.77622
2.5	1.01447	1.14593	1.75679	2.58359	4.46185	5.52098	6.66544	9.22724
3.0	1.02275	1.22976	2.20028	3.53409	6.63581	8.42141	10.3759	14.8313
3.5	1.03541	1.35822	2.88452	5.01272	10.0714	13.0397	16.3274	23.9477
4.0	1.05495	1.55666	3.94789	7.32760	15.5257	20.4209	25.9018	38.7987
4.5	1.08529	1.86528	5.61060	10.9709	24.2181	32.2551	41.3421	63.0208
5.0	1.13269	2.34797	8.22371	16.7304	38.1153	51.2780	66.2936	102.566
5.5	1.20711	3.10654	12.3483	25.8697	60.3939	81.9241	106.685	167.182
6.0	1.32444	4.30354	18.8825	40.4184	96.1907	131.388	172.166	272.838
6.5	1.51007	6.19596	29.2666	63.6416	153.822	211.354	278.457	445.706
7.0	1.80468	9.20931	45.8139	100.799	246.763	340.810	451.182	728.693
7.5	2.27348	14.0029	72.2444	160.373	396.873	550.643	732.133	1192.16
8.0	3.02120	21.6536	114.548	256.061	639.632	891.119	1189.51	1951.53

z\α	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.64872	1.79170	1.93929	2.01485	2.09159	2.24869	2.41069
1.0	2.71828	3.13289	3.57336	3.80361	4.04071	4.53595	5.06016
1.5	4.48169	5.39476	6.39020	6.92028	7.47272	8.64722	9.91880
2.0	7.38906	9.19635	11.2129	12.3045	13.4544	15.9372	18.6789
2.5	12.1825	15.5689	19.4269	21.5461	23.7995	28.7328	34.2760
3.0	20.0855	26.2291	33.3600	37.3286	41.5843	51.0165	61.7801
3.5	33.1155	44.0320	56.9179	64.1737	72.0140	89.5838	109.914
4.0	54.5982	73.7236	96.6443	109.688	123.879	156.001	193.640
4.5	90.0171	123.188	163.492	186.649	212.001	269.909	338.545
5.0	148.413	205.515	275.772	316.494	361.329	464.598	588.290
5.5	244.692	342.432	464.078	535.152	613.814	796.395	1017.20
6.0	403.429	569.984	779.473	902.777	1039.91	1360.45	1751.60
6.5	665.142	947.949	1307.14	1520.00	1757.79	2317.32	3005.76
7.0	1096.63	1575.44	2189.08	2555.04	2965.56	3937.50	5142.69
7.5	1808.04	2616.72	3661.88	4288.89	4994.97	6676.33	8776.41
8.0	2980.96	4343.99	6119.53	7190.61	8401.17	11299.4	14944.4

and

$$B_n = \left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \cdots \frac{1}{\alpha+n-1}\right) - \left(\frac{1}{\gamma} + \frac{1}{\gamma+1} + \cdots \frac{1}{\gamma+n-1}\right) - \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right).$$

Extensive tables of the ψ function are available.*

* H. T. Davis, "Tables of the Higher Mathematical Functions", Vol. I, The Principia Press, Bloomington, Indiana, 1933.

TABLE II: $M(\alpha; 1; z)$

z\α	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00057	1.00571	1.02869	1.05776	1.11703	1.14724	1.17784	1.24021
1.0	1.00132	1.01321	1.06679	1.13540	1.27817	1.35241	1.42859	1.58690
1.5	1.00232	1.02328	1.11839	1.24178	1.50409	1.64328	1.78802	2.09471
2.0	1.00369	1.03705	1.18958	1.39019	1.82607	2.06216	2.31092	2.84820
2.5	1.00558	1.05622	1.28951	1.60066	2.29180	2.67384	3.08162	3.97886
3.0	1.00827	1.08333	1.43200	1.90360	2.97435	3.57807	4.23056	5.69204
3.5	1.01211	1.12226	1.63802	2.34540	3.98620	4.92912	5.96042	8.30961
4.0	1.01769	1.17888	1.93959	2.99716	5.50133	6.96661	8.58729	12.3378
4.5	1.02589	1.26217	2.38583	3.96845	7.78989	10.0641	12.6059	18.5749
5.0	1.03806	1.38594	3.05245	5.42871	11.2729	14.8059	18.7931	28.2840
5.5	1.05628	1.57147	4.05648	7.64094	16.6086	22.1089	28.3718	43.4671
6.0	1.08377	1.85170	5.57965	11.0148	24.8291	33.4155	43.2727	67.3054
6.5	1.12554	2.27781	7.90491	16.1903	37.5574	51.0008	66.5503	104.863
7.0	1.18938	2.92950	11.4742	24.1701	57.3512	78.4616	103.048	164.217
7.5	1.28742	3.93132	16.9795	36.5288	88.2509	121.495	160.459	258.270
8.0	1.43873	5.47833	25.5071	55.7451	136.652	189.144	251.027	407.662

z\α	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.30417	1.36974	1.43695	1.47118	1.50584	1.57642	1.64872
1.0	1.75339	1.92832	2.11198	2.20717	2.30466	2.50666	2.71828
1.5	2.42533	2.78110	3.16328	3.36469	3.57319	4.01219	4.48169
2.0	3.44152	4.09470	4.81173	5.19550	5.59683	6.45439	7.38906
2.5	4.99283	6.13341	7.41115	8.10491	8.83720	10.4235	12.18249
3.0	7.38010	9.31770	11.5295	12.7465	14.0421	16.8839	20.0855
3.5	11.0791	14.3181	18.0805	20.1759	22.4244	27.4133	33.1155
4.0	16.8440	22.2065	28.5359	32.1007	35.9535	44.5924	54.5982
4.5	25.8738	34.6994	45.2706	51.2857	57.8302	72.6467	90.0171
5.0	40.0784	54.5508	72.1215	82.2137	93.2612	118.496	148.413
5.5	62.5213	86.1861	115.294	132.159	150.723	193.479	244.692
6.0	98.0333	136.727	184.838	212.935	244.026	316.176	403.429
6.5	154.467	217.647	297.040	343.748	395.682	517.055	665.142
7.0	244.333	347.456	478.319	555.835	642.410	846.076	1096.63
7.5	387.747	556.046	771.567	900.045	1044.14	1385.20	1808.04
8.0	617.064	891.735	1246.47	1459.19	1698.70	2268.89	2980.96

A more general differential equation whose solution may be expressed in terms of confluent hypergeometric functions is

$$\frac{d^2y}{dx^2} - \frac{a^2x^{2k} + bx^k + c}{(2/k)^2x^2}\, y = 0$$

where

$$y = x^{\frac{1}{2}(1 \pm \sqrt{ck^2+1})}\, e^{-ax^k/2}\, M\left(\frac{b}{4a} + \frac{1}{2}\left(1 \pm \sqrt{c + \frac{1}{k^2}}\right);\right.$$
$$\left. 1 \pm \sqrt{c + \frac{1}{k^2}};\, ax^k.\right) \quad (4)$$

TABLE III: $M(\alpha; \frac{1}{2}; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00037	1.00371	1.01860	1.03740	1.07559	1.09498	1.11457	1.15435
1.0	1.00083	1.00832	1.04197	1.08486	1.17344	1.21916	1.26586	1.36222
1.5	1.00141	1.01418	1.07185	1.14616	1.30238	1.38440	1.46911	1.64683
2.0	1.00216	1.02174	1.11076	1.22677	1.47522	1.60799	1.74669	2.04260
2.5	1.00315	1.03168	1.16231	1.33460	1.71071	1.91537	2.13166	2.60095
3.0	1.00447	1.04498	1.23175	1.48119	2.03654	2.34426	2.67326	3.39906
3.5	1.00625	1.06305	1.32681	1.68356	2.49378	2.95088	3.44519	4.55356
4.0	1.00871	1.08798	1.45884	1.96690	3.14379	3.81957	4.55849	6.24152
4.5	1.01215	1.12287	1.64472	2.36876	4.07876	5.07752	6.18124	8.73307
5.0	1.01702	1.17232	1.90964	2.94540	5.43784	6.91746	8.56909	12.4421
5.5	1.02399	1.24321	2.29139	3.78153	7.43214	9.63277	12.1126	18.0052
6.0	1.03408	1.34590	2.84697	5.00536	10.3832	13.6719	17.4106	26.4052
6.5	1.04881	1.49602	3.66268	6.81170	14.7830	19.7229	25.3851	39.1642
7.0	1.07050	1.71729	4.86981	9.49774	21.3563	28.8452	37.4596	58.6465
7.5	1.10268	2.04584	6.66877	13.5185	31.3560	42.6750	55.8392	88.5356
8.0	1.15075	2.53690	9.36657	19.5732	46.4884	63.7471	83.9493	134.584

$z \backslash \alpha$	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.19496	1.23639	1.27867	1.30013	1.32181	1.36581	1.41069
1.0	1.46265	1.56727	1.67619	1.73231	1.78955	1.90747	2.03008
1.5	1.83603	2.03724	2.25098	2.36271	2.47779	2.71825	2.97293
2.0	2.36445	2.71379	3.09225	3.29293	3.50153	3.94340	4.41972
2.5	3.12228	3.69958	4.33701	4.67964	5.03896	5.81006	6.65520
3.0	4.22221	5.15156	6.19657	6.76558	7.36734	8.67465	10.1300
3.5	5.83596	7.31086	8.99823	9.92826	10.9196	13.0983	15.5592
4.0	8.22631	10.5494	13.2509	14.7569	16.3741	19.9666	24.0800
4.5	11.7973	15.4434	19.7483	22.1740	24.7967	30.6815	37.5051
5.0	17.1722	22.8880	29.7330	33.6282	37.8669	47.4668	58.7290
5.5	25.3164	34.2789	45.1546	51.4005	58.2376	73.8573	92.3820
6.0	37.7301	51.7986	69.0807	79.0905	90.1085	115.483	145.883
6.5	56.7504	78.8684	106.349	122.392	140.141	181.327	231.213
7.0	86.0296	120.865	164.605	190.328	218.923	285.742	367.264
7.5	131.289	186.256	255.955	297.224	343.307	451.696	585.027
8.0	201.510	288.408	399.605	465.867	540.165	715.995	933.960

2. Series Representation. The following series converges absolutely for all values of z.

$$M(\alpha; \gamma; z) = 1 + \frac{\alpha}{\gamma} z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\gamma+n)} \frac{z^n}{n!}. \tag{5}$$

Recurrence relations for M have been tabulated by Jahnke and Emde.†

3. Indefinite Integrals. The confluent hypergeometric function has been used in this laboratory most often as a solution of the differential equation for the

† E. Jahnke and F. Emde, "Tables of Functions", p. 275, Teubner, Leipzig, 1933.

TABLE IV: $M(\alpha; 2; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00027	1.00273	1.01369	1.02751	1.05550	1.06967	1.08397	1.11294
1.0	1.00060	1.00601	1.03025	1.06106	1.12435	1.15685	1.18993	1.25785
1.5	1.00100	1.01000	1.05059	1.10261	1.21111	1.26764	1.32572	1.44671
2.0	1.00149	1.01495	1.07595	1.15490	1.32218	1.41067	1.50252	1.69663
2.5	1.00211	1.02118	1.10809	1.22176	1.46662	1.59822	1.73622	2.03229
3.0	1.00290	1.02915	1.14949	1.30860	1.65736	1.84788	2.04977	2.48953
3.5	1.00392	1.03950	1.20364	1.42314	1.91298	2.18504	2.47638	3.12071
4.0	1.00527	1.05316	1.27558	1.57649	2.26041	2.64661	3.06456	4.00292
4.5	1.00708	1.07145	1.37251	1.78469	2.73886	3.28662	3.88557	5.25020
5.0	1.00953	1.09630	1.50493	2.07108	3.40586	4.18456	5.04464	7.03226
5.5	1.01289	1.13049	1.68812	2.46988	4.34628	5.45813	6.69810	9.60289
6.0	1.01758	1.17811	1.94452	3.03145	5.68592	7.28240	9.07932	13.3435
6.5	1.02416	1.24516	2.30725	3.83038	7.61228	9.91915	12.5383	18.8295
7.0	1.03352	1.34055	2.82543	4.97762	10.4061	13.7614	17.6020	26.9329
7.5	1.04695	1.47747	3.57224	6.63900	14.4892	19.4018	25.0677	38.9800
8.0	1.06636	1.67568	4.65722	9.06341	20.4987	27.7373	36.1450	56.9945

$z \backslash \alpha$	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.14241	1.17238	1.20286	1.21830	1.23386	1.26539	1.29744
1.0	1.32819	1.40100	1.47635	1.51500	1.55430	1.63493	1.71828
1.5	1.57432	1.70881	1.85045	1.92404	1.99953	2.15632	2.32113
2.0	1.90526	2.12919	2.36922	2.49554	2.62620	2.90100	3.19453
2.5	2.35664	2.71117	3.09790	3.30400	3.51894	3.97653	4.47300
3.0	2.98058	3.52742	4.13363	4.46084	4.80484	5.54579	6.36185
3.5	3.85394	4.68461	5.62191	6.13360	6.67570	7.85657	9.17584
4.0	5.09068	6.34434	7.78181	8.57549	9.42253	11.2875	13.3995
4.5	6.86068	8.74766	10.9447	12.1709	13.4887	16.4196	19.7816
5.0	9.41857	12.2588	15.6142	17.5059	19.5523	24.1478	29.4826
5.5	13.1508	17.4299	22.5586	25.4778	28.6551	35.8547	44.3076
6.0	18.6278	25.1013	32.9545	37.4645	42.4019	53.6843	67.0715
6.5	26.7392	36.5564	48.6100	55.5904	63.2735	80.9696	102.176
7.0	38.8235	53.7630	72.3129	83.1393	95.1162	122.906	156.519
7.5	56.9328	79.7471	108.373	125.203	143.909	187.615	240.939
8.0	84.2153	119.176	163.475	189.695	218.969	287.813	372.495

electron velocity distribution function, and in this application the following integrals have been found useful:

$$\int e^{-z} M(\alpha; \gamma; z)\, dz = \frac{e^{-z}}{\alpha - \gamma + 1}\left\{z\frac{d}{dz}[M(\alpha; \gamma; z)] + (\gamma - 1)M(\alpha; \gamma; z)\right\} \quad (6)$$

$$\int ze^{-z} M(\alpha; \gamma; z)\, dz = \frac{e^{-z}}{\alpha - \gamma + 2}\left\{\left(z - \frac{\gamma - 2}{\alpha - \gamma + 1}\right)\left[\left(z\frac{d}{dz}[M(\alpha; \gamma; z)] + (\gamma - 1)M(\alpha; \gamma; z)\right)\right] - zM(\alpha; \gamma; z)\right\}. \quad (7)$$

TABLE V: $W(\alpha; 1; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.3	0.4
0.5	0.451439	0.426196	0.309127	0.150598	−0.214888	−0.663856	−1.23141
1.0	1.89194	1.86292	1.72533	1.53174	1.05899	0.439451	−0.386811
1.5	3.29819	3.26973	3.12913	2.91793	2.35375	1.54433	0.387540
2.0	4.95185	4.92945	4.80806	4.60097	3.96186	2.92708	1.32363
2.5	7.07300	7.06474	6.99538	6.82503	6.13735	4.82556	2.59878
3.0	9.93609	9.95435	9.98798	9.90626	9.21692	7.55704	4.44092
3.5	13.9328	13.9970	14.2141	14.3048	13.6979	11.5942	7.18799
4.0	19.6470	19.7877	20.3161	20.7154	20.3389	17.6644	11.3621
4.5	27.9638	28.2294	29.2731	30.2016	30.3121	26.9006	17.7816
5.0	40.2381	40.7052	42.5895	44.4046	45.4407	41.0782	27.7380
5.5	58.5546	59.3450	62.5895	65.8698	68.5728	62.9906	43.2763
6.0	86.1367	87.4439	92.8790	98.5596	104.173	97.0455	67.6449
6.5	127.986	130.119	139.072	148.663	159.260	150.216	106.014
7.0	191.891	195.339	209.934	225.876	244.894	233.5575	166.629
7.5	290.009	295.555	319.193	345.423	378.547	364.631	262.661
8.0	441.374	450.267	488.397	531.272	587.872	571.384	415.179

$z \backslash \alpha$	0.5	0.6	0.7	0.75	0.8	0.9
0.5	−1.97934*	−3.02899	−4.65969	−5.90077	−7.70160	−16.2842
1.0	−1.52411*	−3.17545	−5.80488	−7.83038	−10.7855	−24.9312
1.5	−1.29260	−3.83568	−8.01278	−11.2835	−16.0953	−39.3270
2.0	−1.14446	−5.04445	−11.6571	−16.9231	−24.7392	−62.8442
2.5	−1.03874	−7.03262	−17.5082	−25.9861	−38.6780	−101.156
3.0	−0.958210*	−10.2132	−26.8475	−40.5111	−61.1308	−163.574
3.5	−0.89413*	−15.2625	−41.7527	−63.8073	−97.3352	−265.344
4.0	−0.84156*	−23.2700	−65.5791	−101.235	−155.803	−431.418
4.5	−0.79738*	−35.9852	−103.745	−161.477	−250.371	−702.648
5.0	−0.75958*	−56.2172	−165.005	−258.616	−403.554	−1145.94
5.5	−0.71351*	−88.4821	−263.525	−415.502	−652.002	−1870.92
6.0	−0.69780*	−140.048	−422.239	−669.251	−1055.43	−3057.25
6.5	−0.6721*	−222.628	−678.321	−1080.20	−1711.18	−4999.49
7.0	−0.6485*	−355.114	−1092.05	−1746.47	−2778.03	−8180.73
7.5	−0.6286*	−568.019	−1761.40	−2827.83	−4515.09	−13393.4
8.0	−0.6089*	−916.663	−2845.35	−4584.43	−7345.43	−21937.6

* Because of the nature of the function for these values, the last figure may be in doubt.

$$\int z^{\gamma} e^{-z(1-\alpha/\gamma)} M(\alpha; \gamma; z) dz$$

$$= \frac{\gamma}{\alpha} \frac{z^{\gamma}}{(1-\alpha/\gamma)} e^{-z(1-(\alpha/\gamma))} \left\{ \frac{d}{dz} [M(\alpha; \gamma; z)] - \frac{\alpha}{\gamma} M(\alpha; \gamma; z) \right\} \quad (8)$$

$$\int z e^{-z(1-\alpha/\gamma)} M(\alpha - \tfrac{1}{2}; \gamma - 1; z)\, dz$$

$$= \frac{\gamma}{\alpha} \frac{e^{-z(1-\alpha/\gamma)}}{(1-\alpha/\gamma)} \left\{ z \frac{d}{dz} [M(\alpha - \tfrac{1}{2}; \gamma - 1; z)] - M(\alpha - \tfrac{1}{2}; \gamma - 1; z) \left[\frac{1}{2} + \frac{\alpha}{\gamma} z \right] \right\}. \quad (9)$$

TABLE VI*: $W(\alpha; 2; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.3
0.5	−2.84756	−2.88691	−3.06956	−3.31745	−3.89302	−4.61032
1.0	−0.826838	−0.860261	−1.01596	−1.22869	−1.72682	−2.35223
1.5	0.309981	0.277995	0.127300	−0.082485	−0.587796	−1.24219
2.0	1.25637	1.22589	1.07957	0.869404	0.339596	−0.381060
2.5	2.19777	2.17015	2.03335	1.82680	1.26970	0.441461
3.0	3.23626	3.21386	3.09601	2.90197	2.32212	1.39899
3.5	4.46240	4.44881	4.36451	4.19757	3.60667	2.54611
4.0	5.98155	5.98207	5.95323	5.83603	5.25551	4.02755
4.5	7.93257	7.95505	8.01442	7.98143	7.44757	6.01544
5.0	10.5091	10.5652	10.7616	10.8654	10.4376	8.75589
5.5	13.9883	14.0953	14.5021	14.8225	14.5963	12.6095
6.0	18.7725	18.9563	19.6836	20.3426	20.4708	18.1113
6.5	25.4515	25.7511	26.9652	28.1487	28.8737	26.0617
7.0	34.8962	35.3699	37.3226	39.3153	41.0201	37.6650
7.5	48.3988	49.1353	52.2086	55.4468	58.7361	54.7409
8.0	67.8872	69.0208	73.7968	78.9507	84.7765	80.0513

$z \backslash \alpha$	0.4	0.5	0.6	0.7	0.75	0.8	0.9
0.5	−5.53635	−6.79061	−8.61092	−11.5497	−13.8519	−17.2595	−33.9886
1.0	−3.16152	−4.25512	−5.83126	−8.34808	−10.3010	−13.1713	−27.1052
1.5	−2.10960	−3.30286	−5.04257	−7.83570	−10.0042	−13.1876	−28.5827
2.0	−1.37344	−2.78062	−4.87872	−8.29916	−10.9730	−14.9089	−33.9700
2.5	−0.713817	−2.44229	−5.09490	−9.51041	−12.9983	−18.1585	−43.2658
3.0	−0.0180700	−2.20138	−5.66247	−11.5612	−16.2785	−23.3013	−57.6938
3.5	0.801770	−2.01904	−6.64562	−14.7269	−21.2736	−31.0861	−79.4984
4.0	1.84251	−1.87504	−8.18483	−19.4780	−28.7451	−42.7303	−112.264
4.5	3.23110	−1.75769	−10.5130	−26.5545	−39.8812	−60.1275	−161.566
5.0	5.14742	−1.65973	−13.9936	−37.0953	−56.5119	−86.1970	−236.030
5.5	7.85487	−1.57484	−19.1848	−52.8478	−81.4477	−125.433	−349.011
6.0	11.7458	−1.50433	−26.9405	−76.4928	−119.013	−184.769	−521.224
6.5	17.4097	−1.44128	−38.5685	−112.154	−175.879	−274.928	−784.887
7.0	25.7385	−1.38542	−56.0751	−166.190	−262.356	−412.539	−1190.24
7.5	38.0870	−1.33574	−82.5480	−248.446	−394.443	−623.450	−1815.78
8.0	56.5221	−1.29082	−122.750	−374.165	−596.998	−947.952	−2784.47

* This table has been computed by the recurrence formula from tables V and VII, therefore there is a possibility that the last number may be off by ± 1.

These integrals are worked out by setting $M(\alpha; \gamma; z) = S(z)/f(z)$, where $f(z)M(\alpha; \gamma; z)$ is the integrand of the required integral. The expression is then differentiated, put into Eq. (1) and the resulting terms integrated by parts. The results may be checked by differentiation and the application of the defining equation in the right hand side of Eqs. (6) to (9).

TABLE VII*: $W(\alpha; 3; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.3
0.5	−9.45028	−9.56093	−10.0789	−10.7932	−12.4992	−14.7097
1.0	−3.54739	−3.60145	−3.85360	−4.19907	−5.01404	−6.05166
1.5	−1.68300	−1.72512	−1.92197	−2.19238	−2.83203	−3.64691
2.0	−0.591663	−0.629037	−0.804794	−1.04882	−1.63504	−2.39427
2.5	0.247797	0.213384	0.049784	−0.181573	−0.752615	−1.54374
3.0	1.00348	0.971889	0.819161	0.597061	0.026498	−0.769050
3.5	1.75745	1.72940	1.59010	1.37873	0.803847	−0.045953
4.0	2.56647	2.54338	2.42309	2.22756	1.64962	0.727450
4.5	3.48293	3.46697	3.37463	3.20380	2.62951	1.61682
5.0	4.56562	4.55999	4.50878	4.37634	3.81880	2.69580
5.5	5.88826	5.89753	5.90655	5.83280	5.31382	4.05796
6.0	7.54887	7.57955	7.67629	7.69094	7.24491	5.83007
6.5	9.68179	9.74321	9.96723	10.1137	9.79362	8.18944
7.0	12.4746	12.5801	12.9885	13.3303	13.2170	11.3886
7.5	16.1922	16.3612	17.0366	17.6666	17.8829	15.7908
8.0	21.2120	21.4724	22.5352	23.5900	24.3217	21.9232

$z \backslash \alpha$	0.4	0.5	0.6	0.7	0.75	0.8	0.9
0.5	−17.6828	−21.8842	−28.2497	−38.9690	−47.6065	−60.6253	−126.177
1.0	−7.42028	−9.31483	−12.1244	−16.7558	−20.4347	−25.9285	−53.2348
1.5	−4.71794	−6.19072	−8.35310	−11.8733	−14.6422	−18.7486	−38.9453
2.0	−3.40247	−4.79827	−6.85123	−10.1849	−12.7967	−16.6563	−35.5145
2.5	−2.54857	−4.00494	−6.17115	−9.70937	−12.4851	−16.5846	−36.5632
3.0	−1.88050	−3.48769	−5.92222	−9.94727	−13.1216	−17.8191	−40.7281
3.5	−1.27858	−3.12059	−5.97664	−10.7772	−14.5938	−20.2630	−47.9994
4.0	−0.671740	−2.84455	−6.30505	−12.2350	−16.9962	−24.1035	−59.0470
4.5	−0.002936	−2.62812	−6.93220	−14.4631	−20.5759	−29.7520	−75.1382
5.0	0.786624	−2.45301	−7.92701	−17.7129	−25.7457	−37.8760	−98.2701
5.5	1.76826	−2.30859	−9.40754	−22.3736	−33.1369	−49.4883	−131.455
6.0	3.03656	−2.18500	−11.5560	−29.0284	−43.6911	−66.0979	−179.187
6.5	4.72289	−2.07949	−14.6452	−38.5414	−58.8040	−89.9440	−248.158
7.0	7.01412	−1.98760	−19.0790	−52.1914	−80.5442	−124.352	−348.355
7.5	10.1798	−1.90670	−25.4551	−71.8746	−111.986	−174.275	−494.721
8.0	14.6125	−1.83475	−34.6591	−100.411	−157.710	−247.113	−709.697

* The last figure in those values near zero may be in doubt because of the rapid rate of change of the function in this region.

ACKNOWLEDGMENT

The tables of the confluent hypergeometric function have been computed by the Joint Computing Group of the Massachusetts Institute of Technology, under the direction of Mrs. Margaret Freeman.

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8. A. H. Heatley, "Transactions of Royal Society of Canada", 1943, defines a function which is a confluent hypergeometric function in z^2 multiplied by a power of z, and gives a short 5-figure table.
9. A. N. Lowan and W. Horenstein, Journal of Mathematics and Physics, **21**, 264, 1942, defines a function $H(m, a, x) = \exp(-ix)M(m + 1 - ia; 2m + 2; 2ix)$, calculates its recurrence relations and gives a 7-figure table for $a = 0(1)10$, $m = 0(1)3$ and $x = 0(1)10$.
10. Applied Mathematics Series 3, (1948), National Bureau of Standards, "Tables of Confluent Hypergeometric Functions $F(\frac{1}{2}n; \frac{1}{2}; x)$ and Related Functions", give 7-figure tables for $x = 0(0.01)0.10$, $n = 3(2)201$, and tables of $\log_e F(\frac{1}{2}n; \frac{1}{2}; x)/\sqrt{2nx}$ for $x = 0.10(0.01)0.60(0.05)2(0.2)7(1)45(5)100$ with $n = 3(2)201$, and $x = 0.6(0.1)2(0.2)7$ with $n = 23(2)201$. This report also contains an expansion of the confluent hypergeometric function in terms of Bessel functions of integral order.

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(Received November 22, 1948)

* The symbol $\alpha = -4(\frac{1}{4})4$ indicates that α-ranges from -4 to 4 at intervals of 1/2.

EVALUATION OF AN INTEGRAL OF A BESSEL FUNCTION

By Wilton R. Abbott

Discussing the response to a unit step function of a semi-infinite artificial transmission line which is mid-series terminated, Gardner and Barnes[1] show that at the n'th terminal the voltage as a function of time is given by the following integral of a Bessel function.

$$v(t, n) = 2n \int_0^t [J_{2n}(\gamma t)]\, dt/t \tag{1}$$

In this relationship $\gamma = 2/(LC)^{1/2}$, L being the series inductance and C the shunt capacitance of the line.

Values of this integral do not seem to have been tabulated. It is possible, however, to expand the integral into a finite series of Bessel functions. Successive integration by parts, for instance, will eventually remove all integral signs from the expression. However, the expansion so obtained is not in general convenient for computation. It is the purpose of this paper to develop two expansions of the integral which will be convenient for computation.

The Lebesgue form of the Laplace Transformation is a convenient medium for carrying out the expansion. The following notation will be used throughout, other notation being introduced as needed. Lower case letters will represent functions of time with the corresponding upper case letters representing the corresponding Laplace Transforms. The complex variable of the Laplace Transformation will be denoted by "s", "t" will represent time, and "n" the number of the terminal at which the voltage is being measured. The symbol $\mathfrak{L}[f(t)]$ is to be read "the Laplace Transform of $f(t)$."

It is well known that

$$\mathfrak{L}[J_n(\gamma t)/t] = \gamma^n/n[s + (s^2 + \gamma^2)^{1/2}]^n. \tag{2}$$

It follows directly from (1) and (2) that

$$V(s, n) = \gamma^{2n}/s[s + (s^2 + \gamma^2)^{1/2}]^{2n}. \tag{3}$$

For convenience let

$$H = \gamma/[s + (s^2 + \gamma^2)^{1/2}] \qquad \text{then} \qquad s = \gamma(1 - H^2)/2H. \tag{4}$$

Then

$$\mathfrak{L}[J_n(\gamma t)/t] = H^n/n \tag{5}$$

and

$$V(s, n) = H^{2n}/s. \tag{6}$$

Now using (4) equation (6) can be expanded into the form (7).

$$V(s, n) = (1/s) - 2(H/\gamma)(1 + H^2 + H^4 + \cdots + H^{2n-2}). \tag{7}$$

[1] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems*, Vol. I (John Wiley and Sons, Inc., New York, 1942), pp. 310 to 317.

Making use of (5) and the known inverse of $1/s$, the inverse transform of (7) is seen to be

$$v(t, n) = 1 - (2/\gamma t)\sum_{k=1}^{n} (2k - 1)J_{2k-1}(\gamma t). \tag{8}$$

Another expansion may be found, this time making use of the Λ functions which are tabulated in Jahnke and Emde[2] through the eighth order, and in reference 4 through the twentieth order.

$$\Lambda_p(\gamma t) = 2^p p! J_p(\gamma t)/(\gamma t)^p. \tag{9}$$

The expansion is facilitated by use of equation 1, section 5:21 of Watson.[3] With ν an integer and $\mu = 1$, this equation becomes

$$J_p(\gamma t) = \frac{(\gamma t)^{p-1}(p-1)!}{2^{p-1}} \sum_{k=0}^{p-1} \frac{(2k+1)}{(p-k-1)!(p+k)!} J_{2k+1}(\gamma t). \tag{10}$$

Using definition (9), equation (11) is obtained from (10).

$$\Lambda_p(\gamma t) = p!(p-1)! \frac{2}{\gamma t} \sum_{k=0}^{p-1} \frac{(2k+1)}{(p-k-1)!(p+k)!} J_{2k+1}(\gamma t) \tag{11}$$

$$\mathfrak{L}[\Lambda_p(\gamma t)] = p!(p-1)! \frac{2H}{\gamma} \sum_{k=0}^{p-1} \frac{H^{2k}}{(p-k-1)!(p+k)!}. \tag{12}$$

Since the transform of Λ is a series of odd powers of H, it is evident that (1) may be expanded as a linear function of the Λ's. To do this $V(s, n)$ may be expanded in the following form where the A's are to be determined.

$$V(s, n) = (1/s) - \sum_{p=0}^{n-1} A_p(n)\mathfrak{L}[\Lambda_{n-p}(\gamma t)]. \tag{13}$$

From (7)
$$V(s, n) = (1/s) - (2H/\gamma)\sum_{j=0}^{n-1} H^{2(n-j-1)}. \tag{14}$$

Equating (13) and (14) leads to the following identity in H.

$$\sum_{p=0}^{n-1} A_p(n)(n-p)!(n-p-1)!\left[\sum_{k=0}^{n-p-1} \frac{H^{2(n-p-k-1)}}{k!(2n-2p-k-1)!}\right] = \sum_{j=0}^{n-1} H^{2(n-j-1)}. \tag{15}$$

Equating like powers of H and solving for the A's leads to the following value of $A_p(n)$.

$$A_p(n) = \frac{(-1)^p(2n-p-1)!}{p!(n-p-1)!(n-p)!}. \tag{16}$$

Substituting this result into (15) it can be shown that corresponding powers of H on the two sides of the equation are equal thus establishing the result that

$$v(t, n) = 1 - \sum_{p=0}^{n-1} \frac{(-1)^p(2n-p-1)!}{p!(n-p-1)!(n-p)!} \Lambda_{n-p}(\gamma t) \tag{17}$$

[2] Jahnke-Emde, *Tables of Functions* (Dover Publications, New York, 1943), pp. 128 and 180 to 189.

[3] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (The MacMillan Company, New York, 1944), p. 139.

The following gives this expansion for values of n up to 9.

$$v(t,1) = 1 - \Lambda_1(\gamma t)$$
$$v(t,2) = 1 + 2\Lambda_1(\gamma t) - 3\Lambda_2(\gamma t)$$
$$v(t,3) = 1 - 3\Lambda_1(\gamma t) + 12\Lambda_2(\gamma t) - 10\Lambda_3(\gamma t)$$
$$v(t,4) = 1 + 4\Lambda_1(\gamma t) - 30\Lambda_2(\gamma t) + 60\Lambda_3(\gamma t) - 35\Lambda_4(\gamma t)$$
$$v(t,5) = 1 - 5\Lambda_1(\gamma t) + 60\Lambda_2(\gamma t) - 210\Lambda_3(\gamma t) + 280\Lambda_4(\gamma t) - 126\Lambda_5(\gamma t)$$
$$v(t,6) = 1 + 6\Lambda_1(\gamma t) - 105\Lambda_2(\gamma t) + 560\Lambda_3(\gamma t) - 1260\Lambda_4(\gamma t) + 126\Lambda_5(\gamma t) - 462\Lambda_6(\gamma t)$$
$$v(t,7) = 1 - 7\Lambda_1(\gamma t) + 168\Lambda_2(\gamma t) - 1260\Lambda_3(\gamma t) + 4200\Lambda_4(\gamma t) - 6930\Lambda_5(\gamma t) + 5544\Lambda_6(\gamma t) - 1716\Lambda_7(\gamma t)$$
$$v(t,8) = 1 + 8\Lambda_1(\gamma t) - 252\Lambda_2(\gamma t) + 2520\Lambda_3(\gamma t) - 11550\Lambda_4(\gamma t) + 27720\Lambda_5(\gamma t) - 36036\Lambda_6(\gamma t) + 24024\Lambda_7(\gamma t) - 6435\Lambda_8(\gamma t)$$
$$v(t,9) = 1 - 9\Lambda_1(\gamma t) + 360\Lambda_2(\gamma t) - 4620\Lambda_3(\gamma t) + 27720\Lambda_4(\gamma t) - 90090\Lambda_5(\gamma t) + 168168\Lambda_6(\gamma t) - 180180\Lambda_7(\gamma t) + 102960\Lambda_8(\gamma t) - 24310\Lambda_9(\gamma t).$$

Values of v of the order of magnitude of unity are found from these equations. With the tables available in footnote [4] the results will be accurate at least to the third place for $n \leq 9$.

Two expansions, equations (8) and (17), have been developed for equation (1). Of these, equation (17) is useful for values of n up to nine. Equation (8) which may involve a little more labor in calculation is useful for values of n up to $(N + 1)/2$ where N is the largest order of Bessel functions for which adequate tables are available. Since the terms in (8) are all of the same sign (except for the first) expansion (8) will be the easier to use for the higher orders.

The author is indebted to Dr. Samuel Silver of the Electrical Engineering Division of the University of California who helped to work out the proof of equation (16).

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(Received October 4, 1948)

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[4] The Mathematical Tables Project, Table of $f_n(x) = \dfrac{n!}{(x/2)^n} J_n(x)$, J. Math. and Phys., 23, pp. 45–60, 1944.

ASYMPTOTIC EXPANSIONS OF SPHEROIDAL WAVE FUNCTIONS

By Milton Abramowitz

The Computation Laboratory of the National Bureau of Standards is now engaged in the computation of the characteristic values b of the spheroidal wave equation

$$(1) \qquad (1 - z^2)y'' - 2(m + 1)zy' + (b - c^2z^2)y = 0$$

for $m = 0(1)4$ and orders $l = 0(1)4$, the parameter c^2 ranging from 0 to the point where the expansion

$$b = nc - \frac{n^2 + 5 + 8m}{2^3} - \frac{n}{64}\frac{(n^2 + 11 - 32m^2)}{c} + \cdots; \quad (n = 2l + 1)$$

will yield satisfactory results.* The solutions have been expressed in terms of the Associated Legendre functions for c small and $0 \leq x \leq 1$.** When $x > 1$ and c small, satisfactory expansions have been obtained in terms of spherical Bessel functions.** When c is large the solution has been expressed in terms of parabolic cylinder functions.† The present article concerns itself with the problem of determining a solution of (1) for the cases when m is large with respect to c and when m and c are both large. If we make the substitution

$$(2) \qquad x = \sqrt{2(m + 1)}z$$

equation (1) becomes

$$(3) \qquad \left\{1 - \frac{x^2}{2(m + 1)}\right\}y'' - xy' + \left\{\frac{b}{2(m + 1)} - \frac{c^2x^2}{4(m + 1)^2}\right\}y = 0$$

If $m \to \infty$, but the ratio $\lambda = b/2(m + 1)$ remains finite then (3) reduces to

$$(4) \qquad y'' - xy' + \lambda y = 0$$

The solutions of this equation for $\lambda = l$ (a positive integer) are the well-known Hermite polynomials.

$$(5) \qquad H_l(x) = (-1)^l e^{x^2/2} d^l(e^{-x^2/2})/dx^l$$

The foregoing suggests the possibility of obtaining a solution of (3) in terms of Hermite polynomials. In the subsequent developments we shall make use of the following functional relations satisfied by the Hermite polynomials.

* This expansion was obtained up to the term c^{-4} by J. Meixner, Neuere Ergebnisse über Sphäroid-Funktionen, Zeitschrift fur Angewandte Mathematik und Mechanik, Bd. 25/27, Heft 5/6, August 1947; the same result was obtained independently by Dr. W. F. Eberlein and communicated to the Computation Labroatory.

** Stratton, Morse, Chu, Hutner; Elliptic Cylinder and Spheroidal Wave Functions, John Wiley & Sons, Inc., New York.

† This solution was obtained by R. Sips of Belgium and independently by Eberlein.

(6a) $$xH_l = H_{l+1} + lH_{l-1}$$

(6b) $$x^2H_l = H_{l+2} + (2l+1)H_l + l(l-1)H_{l-2}$$

(6c) $$H_l'' = l(l-1)H_{l-2}$$

(6d) $$H_l'' - xHl + H_l = 0$$

If we put $\mu = 1/2(m+1)$ equation (3) may be rewritten in the form

(7) $$(1-\mu x^2)y'' - xy' + (\lambda - \mu^2 c^2 x^2)\, y = 0$$

Now assume that y and λ may be expanded in the following manner:

(8) $$y = y_0 + \mu y_1 + \mu^2 y_2 + \cdots$$

(8b) $$\lambda = l + \mu\lambda_1 + \mu^2\lambda_2 + \cdots$$

Substituting (8) into (7) and setting the various powers of μ equal to zero, we get the following system of equations:

(9a) $$y_0'' - xy_0' + ly_0 = 0$$

(9b) $$y_1'' - xy_1' + ly_1 + \lambda_1 y_0 - x^2 y_0'' = 0$$

(9c) $$y_2'' - xy_2' + ly_2 + \lambda_2 y_0 + \lambda y_1 - x^2 y_1'' - c^2 x^2 y_0 = 0$$

(9d) $$y_3'' - xy_3' + ly_3 + \lambda_3 y_0 + \lambda_2 y_1 + \lambda_1 y_2 - x^2 y_2'' - c^2 x^2 y_1 = 0$$

The form of the further equations is apparent from the above. The solution of (9a) is

(10) $$y_0 = H_l(x)$$

Equation (9b) now becomes

(11) $$y_1'' - xy_1' + ly_1 + \lambda_1 H_l - x^2 H_l'' = 0$$

However, with the aid of (6c) and (6b), equation (11) becomes

(12) $$y_1'' - xy_1' + ly_1 + \{\lambda_1 - l(l-1)\}\, H_l - l(l-1)(2l-3)H_{l-2} - l(l-1)(l-2)(l-3)H_{l-4} = 0$$

In order to determine the value of λ_1 and solve equation (12) we make the following observations. The solution of (12) may be assumed to be of the form $y_1 = \Sigma\, a_k^{(l)} H_k(x)$. In this expansion if $k = l$, $H_l(x)$ will be annihilated by the first three terms in (12). Therefore the solution will not contain $a_l^{(l)} H_l(x)$. Furthermore since in (12) the coefficients of $H_k(x)$ must be set equal to zero for each k this will require that the coefficient of $H_l(x)$ be zero and therefore we get (13) $\lambda_1 = l(l-1)$.

If $k \neq l-2$ or $l-4$, substitution of $a_k^{(l)} H_k(x)$ in (12) will produce a term of the form $(l-k)a_k^{(l)} H_k(x)$ and therefore since $(l-k) \neq 0$, it follows that $a_k^{(l)} = 0$. Thus the expression for y_1 must be of the form

(14) $$y_1 = a_{l-2}^{(l)} H_{l-2}(x) + a_{l-4}^{(l)} H_{l-4}(x).$$

Substituting in (12) and setting the coefficients of H_{l-2} and H_{l-4} equal to zero we find

$$(15) \qquad a_{l-2}^{(1)} = \tfrac{1}{2}l(l-1)(2l-3); \qquad a_{l-4}^{(1)} = \tfrac{1}{4}l(l-1)(l-2)(l-3)$$

The method just described may be employed to determine the solutions of (9c), (9d), $\cdots$ It is then found that

$$(16) \qquad \lambda_2 = (2l+1)c^2$$

and

$$(17) \qquad y_2 = a_{l+2}^{(2)}H_{l+2} + a_{l-2}^{(2)}H_{l-2} + a_{l-4}^{(2)}H_{l-4} + a_{l-6}^{(2)}H_{l-6} + a_{l-8}^{(2)}H_{l-8}$$

where

$$a_{l+2}^{(2)} = -c^2/2, \qquad a_{l-2}^{(2)} = \tfrac{1}{2}[l(l-1)c^2 - 2(2l-3)a_{l-2}^{(1)}]$$

$$a_{l-4}^{(2)} = \tfrac{1}{4}[(l-2)(l-3)(2l-7)a_{l-2}^{(1)} - 4(2l-5)a_{l-4}^{(1)}]$$

$$a_{l-6}^{(2)} = \tfrac{1}{6}[(l-4)(l-5)(2l-11)a_{l-4}^{(1)} + (l-2)(l-3)(l-4)(l-5)a_{l-6}^{(1)}]$$

$$a_{l-8}^{(2)} = \tfrac{1}{8}[(l-4)(l-5)(l-6)(l-7)a_{l-4}^{(1)}]$$

Further $\lambda_3 = -c^2(6l^2 + 2l + 1)$; the corresponding expression for y_3 can be obtained in a similar way but will not be given here. With the value of λ_1, λ_2 and λ_3 above obtained we get

$$(18) \qquad \begin{aligned} b &= 2(m+1)\lambda \\ &= 2(m+1)l + l(l-1) + \frac{(2l+1)c^2}{2(m+1)} - \frac{c^2(6l^2+2l+1)}{2(m+1)^2} + \cdots \end{aligned}$$

The accuracy of (18) may be judged from the following two examples:

(1) $m = 3, l = 0, c^2 = 1$ From (18) $b = .1094$
True value $b = .11013$

(2) $m = 100, l = 2, c^2 = 1$ From (18) $b = 406.02475$
True value $b = 406.02406$

An alternative derivation of (18) is as follows: If we assume that the solution of (7) may be expressed in the form

$$(19) \qquad y = \Sigma A_k H_k(x)$$

and substitute in (7) we obtain the following four-term recursion formula

$$\alpha A_k - \beta A_{k+2} - \delta A_{k+4} - c^2\mu^2 A_{k-2} = 0$$

$$(20) \qquad \begin{aligned} \alpha &= \lambda - k - \mu k(k-1) - (2k+1)c^2\mu^2 \\ \beta &= \mu(k+1)(k+2)(2k+1) + c^2\mu^2(k+1)(k+2) \\ \delta &= \mu(k+1)(k+2)(k+3)(k+4) \end{aligned}$$

If we now assume that

$$\lambda = l + \lambda_1\mu + \lambda_2\mu^2 + \cdots$$

and that the coefficients A_k are such that

$$\text{if } k = l, \qquad A_k = 1$$

$$\text{if } k \neq l, \qquad A_k = a_1^{(k)}\mu + a_2^{(k)}\mu + \cdots$$

substitution in (20) will ultimately yield the values of $\lambda_1, \lambda_2, \cdots$ and the values of $a_1^{(k)}, a_2^{(k)}, \cdots$

Let us assume now that both m and c become large in such a manner $[c/2(m+1)]^2 = \delta$ remains finite, equation (7) then becomes in the limit

$$y'' - xy' + (\lambda - \delta x^2)y = 0 \tag{21}$$

This equation may be reduced to Weber's equation by the transformations $y = e^{s^2/4}f(x)$, $2kx = s$ where $k^4 = \frac{1}{4}(\delta + \frac{1}{4})$. This procedure suggests that we make a similar transformation in equation (7) and obtain

$$(1 - \mu x^2)f'' - \mu x^3 f' + \{(\lambda + \tfrac{1}{2}) - (\delta + \tfrac{1}{4})x^2 - \mu(\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4)\}\, y = 0 \tag{22}$$

In this last equation we again put $2kx = s$ and obtain

$$\left(1 - \frac{\mu s^2}{4k^2}\right)f'' - \frac{\mu s^3}{16k^4}f' + \left\{\frac{\lambda + \frac{1}{2}}{4k^2} - \frac{s^2}{4} - \mu\left(\frac{s^2}{32k^4} - \frac{s^4}{256k^8}\right)\right\}f = 0 \tag{23}$$

To solve equation (23) we now assume

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \cdots \tag{24a}$$

$$(\lambda + \tfrac{1}{2})/4k^2 = \lambda_0 + \mu\lambda_1 + \mu^2\lambda_2 + \cdots \tag{24b}$$

Substituting in (23) we get a system of equations to be solved successively. We write only the first and second equations of the system, namely

$$f''_0 + \{\lambda_0 - \tfrac{1}{4}s^2\}\, f_0 = 0 \tag{25a}$$

$$f''_1 + \left\{\lambda_0 - \frac{s^2}{4}\right\}f_1 - \frac{s^2}{4k^2}f''_0 - \frac{s^3}{16k^4}f'_0 + \lambda_1 f_0 - \frac{s^2}{32k^4}f_0 - \frac{s^4}{256k^8}f_0 = 0 \tag{25b}$$

It is well known that (25a) has a solution which is finite over the entire range of s if $\lambda_0 = l + \frac{1}{2}$ in which case

$$f_0(s) = D_l(s) = (-1)^l e^{s^2/4}d^l(e^{-s^2/2})/ds^l = 2^{-l/2}e^{-s^2/4}H_l(s/\sqrt{2}) \tag{26}$$

The functions $D_l(s)$ are the parabolic cylinder functions.

We shall not give the solution of (25b) here but shall determine the value of λ_1 in a similar manner to that employed for equation (9b). For this purpose the following relations are useful.

$$D_l'' + \{l + \tfrac{1}{2} - s^2/4\}\, D_l = 0 \tag{27a}$$

$$D_l' = D_{l-1} - (s/2)D_l \tag{27b}$$

$$sD_l = D_{l+1} + lD_{l-1} \tag{27c}$$

$$s^2 D_l = D_{l+2} + (2l + 1)D_l + l(l - 1)D_{l-2} \tag{27d}$$

(27e) $s^3D_l = D_{l+3} + 3(l+1)D_{l+1} + 3l^2D_{l-1} + l(l-1)(l-2)D_{l-3}$

(27f) $s^4D_l = D_{l+4} + (4l+6)D_{l+2} + (6l^2+6l+3)D_l + 2l(l-1)(2l-1)D_{l-2} + l(l-1)(l-2)(l-3)D_{l-4}$

To determine the value of λ_1 we need only find the coefficient of D_l in (25b) and set it equal to zero. We then find

$$\lambda_1 = \frac{1-2l-2l^2}{16k^2} - \frac{(2l+1)}{16k^4} + \frac{6l^2+6l+3}{256k^6} \tag{28}$$

The approximation for b in terms of the origin al parameters thus becomes

$$b = (2l+1)\sqrt{c^2+(m+1)^2} - (m+1) + \tfrac{1}{4}(1-2l-2l^2) - \frac{(2l+1)(m+1)}{\sqrt{c^2+(m+1)^2}} + \frac{(6l^2+6l+3)(m+1)^2}{4[c^2+(m+1)^2]} + \cdots \tag{29}$$

The accuracy of (29) may be judged by the following example. For the case $m = 100$, $l = 2$, $c = 100$ the true value of b obtained by the continued fraction method is $b = 608.2993$; the value obtained by (29) is $b = 608.2715$ while (18) yields 646.4.

It should be noted that (29) may also be used for small values of c. Thus in the example cited previously for $m = 100$, $l = 2$, $c^2 = 1$, we find $b = 406.0237$ as compared with the true value $b = 406.02406$ obtained by the continued fraction method.

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(Received September 27, 1948)

FORMULAS FOR COMPLEX CARTESIAN INTERPOLATION OF HIGHER DEGREE

By Herbert E. Salzer

In interpolating for an analytic function tabulated over a square grid in the complex plane, according to Lagrange's formula, the use of explicit Lagrangian coefficients is feasible when the interpolation polynomial is based upon no more than seven points and in the interval $P = p + iq$ (where $f(z) = f(z_0 + Ph)$ and h is the length of the sides of the squares), neither p nor q is finer than 0.1. The reason is that Lagrangian coefficients for that purpose have already been tabulated.* But for the cases where $f(z)$ is required when p or q has more than one decimal place, or where the interpolation polynomial must be based upon eight or nine points (i.e. 7th or 8th degree interpolation), the calculation of the separate Lagrangian coefficients, which are polynomials in the complex argument P, involves a prohibitive amount of labor. Fortunately a much shorter method of interpolation is available, because the scheme which was originated by W. J. Taylor** for real Lagrangian interpolation where the argument is at equal intervals, can be generalized for complex Lagrangian interpolation based upon any set of fixed points. If A_k denotes $1/\pi'(z_k - z_j)$, where π' means the product for each of the fixed points z_j besides z_k, then the Lagrangian interpolation polynomial is given by $\Sigma\, a_k f_k / \Sigma\, a_k$, where $a_k \equiv A_k/(z - z_k)$, $f_k \equiv f(z_k)$, and the summation is over all points z_k.

In complex Cartesian interpolation for analytic functions it is simplest to make P the variable, so that $A_k \equiv 1/\pi'(k - j)$ and $a_k \equiv A_k/(P - k)$ where k (also j) is now a small complex integer. Thus $k = 0$ corresponds to z_0, $k = 1 + 2i$ corresponds to $z_0 + h + 2ih$, etc. For different configurations of the fixed points, as well as different numbers of points, the quantities A_k and a_k will vary, but for the sake of avoiding too much notation, the particular degree and configuration will be understood wherever A_k is given. The coefficients A_k are given here for configurations in the z_k's based upon three points through nine points (i.e. complex quadratic through octic interpolation). Corresponding to each of the first configurations for the five- to nine-point cases (the first configuration is applicable anywhere in a table, including the beginning, end, or corners) a second alternative configuration, also applicable anywhere in a Cartesian table, is given for checking purposes. Furthermore, in the five-through nine-point cases there are given additional configurations which have the property of making the location of P (or z) considerably more central with regard to the points k (or z_k), and hence would be expected to yield greater accuracy. The number of these

* A. N. Lowan, H. E. Salzer, "Coefficients for Interpolation within a Square Grid in the Complex Plane", *Jour. of Math. & Phys. 23*, pp. 156–166; H. E. Salzer "Coefficients for Complex Quartic, Quintic, and Sextic Interpolation within a Square Grid," *ibid. 27*, 1948 pp. 136–156.

** "Method of Lagrangian Curvilinear Interpolation," *Jour. Res. Nat. Bur. Stand 35*, 1945, RP 1667, pp. 151–155.

more central configurations which are given are: for five-point—one, for six- and seven-point—two, for eight- and nine-point—three. Thus the user has a great latitude of choice in available formulas for complex interpolation, for checking that interpolation, and for central choice of the argument P.

In addition to their use in interpolation, the quantities A_k which are given below, have two other important applications:

a) For checking of complex functions tabulated at the n-points in any particular configuration, since, except for a trivial factor of h^{-n+1}, A_k is also the coefficient of f_k in the complex divided difference of the $(n-1)$th order, which is vanishingly small for a function behaving as a polynomial in z (or P) of the $(n-2)$th degree. As a special application of a), one may accomplish

b) Complex extrapolation, so that for any n-point configuration, if a function behaves as a polynomial in z (or P) of the $(n-2)$th degree and is known at any $(n-1)$ of those points, its value at the nth point can be obtained readily.

Thanks are due to Mr. Samuel Zuckerman for assistance in the checking of the A_k's.

TABLE OF COEFFICIENTS A_k

Three-Point Interpolation (Quadratic)

Configuration	*Coefficients*
$\dot{i}$ $\dot{0}\ \dot{1}$	$A_0 = -i,\ A_1 = \frac{1}{2}(1+i),\ A_i = \frac{1}{2}(-1+i).$

Four-Point Interpolation (Cubic)

Configuration	*Coefficients*
$\dot{i}\ \ 1+\dot{i}$ $\dot{0}\ \ \dot{1}$	$A_0 = \frac{1}{2}(1+i),\ A_1 = \frac{1}{2}(-1+i),\ A_i = \frac{1}{2}(1-i),$ $A_{1+i} = \frac{1}{2}(-1-i).$

Five-Point Interpolation (Quartic)

Configuration	*Coefficients*
$i\ \ \dot{1}+i$ $\dot{0}\ \ \dot{1}\ \ \ \dot{2}$	$A_0 = \frac{1}{2}(-1-i),\ A_1 = \frac{1}{2}(1-i),\ A_2 = \frac{1}{10}(1+3i),$ $A_i = \frac{1}{10}(-3+i),\ A_{1+i} = \frac{1}{2}i.$
$\dot{i}\ \ \dot{1}+i\ \ \dot{2}+i$ $\dot{0}\ \ \dot{1}$	$A_0 = \frac{1}{10}(-3-i),\ A_1 = -\frac{1}{2}i,\ A_i = \frac{1}{2}(-1+i),$ $A_{1+i} = \frac{1}{2}(1+i),\ A_{2+i} = \frac{1}{10}(1-3i).$
$\dot{2}+2i$ $\dot{i}\ \ 1+\dot{i}$ $\dot{0}\ \ \dot{1}$	$A_0 = -\frac{1}{2},\ A_1 = \frac{1}{10}(-1-3i),\ A_i = \frac{1}{10}(-1+3i),$ $A_{1+i} = \frac{1}{2},\ A_{2+2i} = -\frac{1}{10}.$

Six-Point Interpolation (Quintic)

Configuration	*Coefficients*
$\dot{2}i$ $\dot{i}\ \ \ 1+\dot{i}$ $\dot{0}\ \ \ \dot{1}\ \ \ \dot{2}$	$A_0 = \frac{1}{2}(1-i),\ A_1 = \frac{1}{10}(3+i),\ A_2 = \frac{1}{10}(-1+2i),$ $A_i = \frac{1}{10}(-1-3i),\ A_{1+i} = \frac{1}{2}(-1+i),\ A_{2i} = \frac{1}{10}(-2+i).$
$\dot{i}\ \ \ 1+\dot{i}\ \ \dot{2}+i$ $\dot{0}\ \ \ \dot{1}\ \ \ \dot{2}$	$A_0 = \frac{1}{10}(3+i),\ A_1 = \frac{1}{2}i,\ A_2 = \frac{1}{10}(-3+i),$ $A_i = \frac{1}{10}(3-i),\ A_{1+i} = -\frac{1}{2}i,\ A_{2+i} = \frac{1}{10}(-3-i).$
$\dot{i}\ \ \ 1+\dot{i}$ $-1\ \ \dot{0}\ \ \ \dot{1}\ \ \ \dot{2}$	$A_{-1} = \frac{1}{10}(-1+3i),\ A_0 = \frac{1}{2}(-1-i),\ A_1 = \frac{1}{2}(1-i),$ $A_2 = \frac{1}{10}(1+3i),\ A_i = \frac{1}{10}(-1+2i),\ A_{1+i} = \frac{1}{10}(1+2i).$
$-1+i\ \ \dot{i}\ \ \ 1+\dot{i}$ $\ \ \ \ \dot{0}\ \ \ \dot{1}\ \ \ \dot{2}$	$A_0 = -\frac{1}{2}i,\ A_1 = \frac{1}{10}(3-i),\ A_2 = \frac{1}{10}i,$ $A_{-1+i} = -\frac{1}{10}i,\ A_i = \frac{1}{10}(-3+i),\ A_{1+i} = \frac{1}{2}i,$

TABLE OF COEFFICIENTS A_i—*Continued*

Seven-Point Interpolation (Sextic)

Configuration	*Coefficients*
$\dot{2}i$ $\dot{i}\quad 1+\dot{i}\quad \dot{2}+i$ $\dot{0}\quad 1\quad \dot{2}$	$A_0 = \frac{1}{10}(-1+3i)$, $A_1 = \frac{1}{10}(-2+i)$, $A_2 = \frac{1}{10}(-2-i)$, $A_i = \frac{1}{10}(1+3i)$, $A_{1+i} = \frac{1}{2}(1-i)$, $A_{2+i} = \frac{1}{10}(-1-i)$, $A_{2i} = \frac{1}{10}$.
$\dot{2}i\quad\quad 2+\dot{2}i$ $\dot{i}\quad 1+\dot{i}$ $\dot{0}\quad 1\quad 2$	$A_0 = \frac{1}{10}i$, $A_1 = \frac{1}{10}(-1+i)$, $A_2 = \frac{1}{10}(-2-i)$, $A_i = \frac{1}{10}(1+i)$, $A_{1+i} = -\frac{1}{2}i$, $A_{2i} = \frac{1}{10}(2-i)$, $A_{2+2i} = \frac{1}{10}i$.
$\dot{2}i$ $\dot{i}\quad\quad 1+\dot{i}$ $-1\quad \dot{0}\quad \dot{1}\quad\quad \dot{2}$	$A_{-1} = \frac{1}{10}(-1-i)$, $A_0 = \frac{1}{2}(1-i)$, $A_1 = \frac{1}{10}(3+i)$, $A_2 = \frac{1}{110}(-1+2i)$, $A_i = \frac{1}{10}(-2-i)$, $A_{1+i} = \frac{1}{10}(-1+3i)$, $A_{2i} = \frac{1}{10}i$.
$\dot{i}\quad\quad 1+\dot{i}\quad 2+\dot{i}$ $-1\quad \dot{0}\quad 1\quad\quad 2$	$A_{-1} = -\frac{1}{10}i$, $A_0 = \frac{1}{10}(3+i)$, $A_1 = \frac{1}{2}i$, $A_2 = \frac{1}{10}(-3+i)$, $A_i = \frac{1}{10}(1-2i)$, $A_{1+i} = \frac{1}{10}(-1-2i)$, $A_{2+i} = -\frac{1}{10}$.

Eight-Point Interpolation (Septic)

Configuration	*Coefficients*
$\dot{2}i\quad 1+\dot{2}i$ $\dot{i}\quad 1+\dot{i}\quad 2+\dot{i}$ $\dot{0}\quad 1\quad 2$	$A_0 = \frac{1}{10}(-1-i)$, $A_1 = \frac{1}{10}(-1-2i)$, $A_2 = -\frac{1}{10}i$, $A_i = \frac{1}{10}(-2-i)$, $A_{1+i} = \frac{1}{2}(1+i)$, $A_{2+i} = -\frac{1}{110}i$, $A_{2i} = -\frac{1}{10}$, $A_{1+2i} = -\frac{1}{10}$.
$\dot{2}i\quad\quad \dot{2}+2i$ $\dot{i}\quad 1+\dot{i}\quad 2+i$ $\dot{0}\quad 1\quad \dot{2}$	$A_0 = \frac{1}{10}(-1-2i)$, $A_1 = -\frac{1}{10}i$, $A_2 = \frac{1}{10}(1-2i)$, $A_i = \frac{1}{10}(-1-i)$, $A_{1+i} = \frac{1}{2}i$, $A_{2+i} = \frac{1}{10}(1-i)$, $A_{2i} = -\frac{1}{10}$, $A_{1+2i} = \frac{1}{10}$.
$-1+i\quad \dot{i}\quad\quad 1+i\quad 2+\dot{i}$ $-1\quad\quad \dot{0}\quad\quad \dot{1}\quad\quad 2$	$A_{-1} = \frac{1}{10}$, $A_0 = \frac{1}{10}(1+2i)$, $A_1 = \frac{1}{10}(-1+2i)$, $A_2 = -\frac{1}{10}$, $A_{-1+i} = \frac{1}{10}$, $A_i = \frac{1}{10}(1-2i)$, $A_{1+i} = \frac{1}{10}(-1-2i)$, $A_{2+i} = -\frac{1}{10}$.
$\dot{i}\quad\quad 1+i$ $-1\quad \dot{0}\quad\quad \dot{1}\quad\quad \dot{2}$ $\quad -i\quad\quad 1-i$	$A_{-i} = \frac{1}{10}i$, $A_{1-i} = \frac{1}{10}i$, $A_{-1} = -\frac{1}{10}$, $A_0 = \frac{1}{2}$, $A_1 = -\frac{1}{2}$, $A_2 = \frac{1}{10}$, $A_i = -\frac{1}{10}i$, $A_{1+i} = -\frac{1}{10}i$.
$-1+2i\quad\quad\quad\quad \dot{2}+2i$ $\quad\quad \dot{i}\quad 1+i$ $\quad\quad \dot{0}\quad \dot{1}$ $-1-i\quad\quad\quad\quad \dot{2}-i$	$A_{-1-i} = \frac{1}{110}(-1-i)$, $A_{2-i} = \frac{1}{110}(1-i)$, $A_0 = \frac{1}{10}(1+i)$, $A_1 = \frac{1}{10}(-1+i)$, $A_i = \frac{1}{10}(1-i)$, $A_{1+i} = \frac{1}{10}(-1-i)$, $A_{-1+2i} = \frac{1}{110}(-1+i)$, $A_{2+2i} = \frac{1}{110}(1+i)$.

Nine-Point Interpolation (Octic)

Configuration	*Coefficients*
$\dot{2}i\quad 1+\dot{2}i\quad \dot{2}+2i$ $\dot{i}\quad 1+\dot{i}\quad \dot{2}+i$ $\dot{0}\quad 1\quad \dot{2}$	$A_0 = \frac{1}{10}$, $A_1 = \frac{1}{10}$, $A_2 = \frac{1}{10}$, $A_i = \frac{1}{10}$, $A_{1+i} = -\frac{1}{2}$, $A_{2+i} = \frac{1}{10}$, $A_{2i} = \frac{1}{10}$, $A_{1+2i} = \frac{1}{10}$, $A_{2+2i} = \frac{1}{10}$.
$\dot{2}i\quad 1+\dot{2}i$ $\dot{i}\quad 1+\dot{i}\quad \dot{2}+i$ $\dot{0}\quad \dot{1}\quad\quad \dot{2}\quad\quad \dot{3}$	$A_0 = \frac{1}{110}(1+i)$, $A_1 = \frac{1}{10}(1+2i)$, $A_2 = \frac{1}{10}i$, $A_3 = \frac{1}{1130}(-5+i)$, $A_i = \frac{1}{10}(1+i)$, $A_{1+i} = \frac{1}{10}(-1-3i)$, $A_{2+i} = \frac{1}{10}(-1+i)$, $A_{2i} = \frac{1}{110}(3+2i)$, $A_{1+2i} = \frac{1}{10}(1+i)$.

	$\dot{2}i$			$A_{-i} = \frac{1}{130}(-1+2i)$, $A_0 = \frac{1}{10}(-2+i)$, $A_1 = -\frac{1}{10}$,
$-\dot{1}+i$	$\dot{i}$	$\dot{1}+i$	$\dot{2}+i$	$A_i = \frac{1}{130}(-1-i)$, $A_{-1+i} = \frac{1}{130}(-1+i)$, $A_{1+i} = \frac{1}{10}(2+i)$,
$-\dot{1}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$A_{1+i} = \frac{1}{10}(1-3i)$, $A_{2+i} = \frac{1}{130}(-2-i)$, $A_{2i} = \frac{1}{130}(-1-3i)$.
	$\dot{3}i$			$A_{-i} = \frac{1}{1300}(-1+3i)$, $A_0 = \frac{1}{130}(-3+i)$, $A_1 = -\frac{1}{10}$,
	$\dot{2}i$	$1+\dot{2}i$		$A_i = \frac{1}{1300}(-2-3i)$, $A_{1+i} = \frac{1}{10}(1+3i)$, $A_{2i} = \frac{1}{130}(-1-7i)$,
	$\dot{i}$	$1+\dot{i}$		$A_{2i} = \frac{1}{10}$, $A_{1+2i} = \frac{1}{10}(1-i)$,
$-\dot{1}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$A_{2i} = \frac{1}{7300}(-3-11i)$.
	$2\dot{i}$			$A_{-i} = -\frac{1}{10}$, $A_{1-i} = \frac{1}{130}(-3+i)$, $A_{-1} = \frac{1}{130}(1-2i)$,
	$\dot{i}$	$1+\dot{i}$		$A_0 = \frac{1}{10}i$, $A_1 = \frac{1}{10}(-1-2i)$, $A_2 = \frac{1}{130}(1+i)$,
$-\dot{1}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$A_i = \frac{1}{10}$, $A_{1+i} = \frac{1}{10}(1-i)$, $A_{2i} = \frac{1}{1300}(-1-3i)$.
	$-\dot{i}$	$1-\dot{i}$		

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(Received October 15, 1948)

APPLICATION OF THE W.K.B. METHOD TO THE FLOW OF A COMPRESSIBLE FLUID, II

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1. Introduction. In a previous paper [9] one of the present authors has expressed the fundamental equations of motion for a two-dimensional compressible fluid flow in terms of new variables suggested by the W.K.B. method, and obtained a new method of approximation for treating the subsonic flow past arbitrary profiles. In this paper the method will be applied to some typical cases of interest in order to test its usefulness.

2. Outline of the new method. For convenience a brief account will be given of the method of approximation put forward in the previous paper.

The flow of an incompressible fluid past a profile P_0 in the ζ-plane corresponds to the flow of a compressible fluid past a profile P in the z-plane, where the complex coordinates z_P and ζ_P of the profiles P and P_0 are related by the equation:

$$dz_P = Q/q\, K^{\frac{1}{2}}\, d\zeta_P, \tag{2.1}$$

and q and Q denote the magnitudes of velocity at corresponding points in the z- and ζ-plane respectively. Also Q and K are functions of q defined as follows:

$$Q = Ce^{-\tau}, \qquad K = \mu^2(\rho_0/\rho)^2, \tag{2.2,3}$$

with

$$\tau = -\int_1^q \mu\, dq/q, \qquad \mu = [1 - (q/c)^2]^{\frac{1}{2}}, \tag{2.4,5}$$

where C is a certain constant, ρ the density, ρ_0 the stagnation density, and c is the local velocity of sound. For a gas obeying the adiabatic law we have

$$K = (1 - q^2)(1 - a^2q^2)^{-1/a^2}, \tag{2.6}$$

$$\mu = \left(\frac{1 - q^2}{1 - a^2q^2}\right)^{\frac{1}{2}}, \qquad a^2 = \frac{\gamma - 1}{\gamma + 1}, \tag{2.7}$$

$$\begin{aligned} Q &= C\left(\frac{1 + a\mu}{1 - a\mu}\right)^{1/2a}\left(\frac{1 - \mu}{1 + \mu}\right)^{\frac{1}{2}} \\ &= \frac{2}{1 + \mu}\left(\frac{1 + a\mu}{1 + a}\right)^{1/a} q(1 - a^2q^2)^{(1-a)/2a}, \end{aligned} \tag{2.8}$$

where the constant C has been determined so that $Q/q \to 1$ as $q \to 0$. γ is of course the ratio of the specific heats. Taking $\gamma = 1.4$ for air we have prepared the numerical tables of Q, $Qq^{-1}K^{\frac{1}{2}}$, which are given in Table 1.

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TABLE 1

q	Q	$Q/q\,K^{\frac{1}{2}}$	q	Q	$Q/q\,K^{\frac{1}{2}}$
0.00	0.0000	1.0000	0.50	0.4734	0.9391
1	100	1.0000	51	4817	9362
2	200	0.9999	52	4900	9332
3	300	9998	53	4982	9301
4	400	9996	54	5063	9269
5	500	9994	55	5144	9236
6	600	9992	56	5223	9202
7	699	9990	57	5302	9166
8	799	9987	58	5381	9130
9	899	9983	59	5458	9092
0.10	998	9979	0.60	5535	9053
11	1097	9974	61	5610	9012
12	1196	9970	62	5685	8970
13	1295	9964	63	5759	8926
14	1394	9958	64	5832	8881
15	1493	9953	65	5904	8835
16	1591	9946	66	5975	8786
17	1690	9939	67	6045	8736
18	1788	9932	68	6114	8683
19	1886	9924	69	6182	8629
0.20	1983	9915	0.70	6250	8573
21	2081	9906	71	6315	8514
22	2178	9896	72	6380	8454
23	2275	9886	73	6444	8390
24	2371	9876	74	6506	8324
25	2467	9865	75	6568	8256
26	2563	9854	76	6628	8184
27	2659	9841	77	6687	8109
28	2754	9829	78	6745	8031
29	2849	9816	79	6801	7949
0.30	2943	9802	0.80	6856	7863
31	3037	9788	81	6910	7773
32	3131	9773	82	6962	7677
33	3224	9758	83	7013	7577
34	3317	9742	84	7063	7472
35	3410	9726	85	7111	7360
36	3502	9708	86	7157	7242
37	3593	9690	87	7201	7115
38	3684	9671	88	7244	6980
39	3775	9652	89	7285	6835
0.40	3865	9633	0.90	7324	6678
41	3954	9612	91	7361	6508
42	4043	9591	92	7396	6322
43	4132	9569	93	7429	6117
44	4220	9546	94	7460	5886
45	4307	9522	95	7488	5624
46	4393	9498	96	7513	5318
47	4479	9472	97	7535	4947
48	4565	9446	98	7554	4467
49	4650	9419	99	7569	3752
0.50	0.4734	0.9391	1.00	0.7576	0.0000

Putting $z_P = x + iy$, $\zeta_P = \xi + i\eta$, (2.1) can be written as

$$dx = (Q/q)K^{\frac{1}{2}}\,d\xi, \qquad dy = (Q/q)K^{\frac{1}{2}}\,d\eta. \tag{2.9}$$

For any given profile P_0 in the ζ-plane the velocity Q on the surface may be assumed to be known (theoretically or experimentally). Then q and hence $Qq^{-1}K^{\frac{1}{2}}$ will be found with the aid of Table 1. Next, by means of (2.9), the form of the profile P in the physical z-plane will be obtained by numerical or graphical integration.

Further, the distribution of pressure p can be evaluated by the formula

$$p/p_0 = (1 - a^2q^2)^{\gamma/(\gamma-1)}. \tag{2.10}$$

It may be remarked that the numerical tables of p/p_0, ρ/ρ_0, q/c, μ, $q\rho/\rho_0$ have already been prepared and conveniently employed by the senior author [7].

3. Flat plate accompanied with dead air. The discontinuous flow of an incompressible fluid impinging normally on a flat plate, leaving a dead water region behind it, is well known as Kirchhoff's problem. We can easily find that the velocity Q at any point (ξ, η) on the front surface of the flat plate is given by

$$\frac{Q}{Q_\infty} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right), \tag{3.1}$$

where

$$\xi = 0, \qquad \eta = \frac{l_0}{2}\frac{1}{4+\pi}(\sin 2\theta + 4\cos\theta - 2\theta + \pi). \tag{3.2}$$

Here l_0 is the width of the plate and Q_∞ is the undisturbed velocity at infinity.

For the numerical calculation, it proves to be sufficiently accurate to take only 11 values for θ: $\theta_N = N\pi/20 (N = 0, 1, 2, \cdots, 10)$. We first find Q/Q_∞ by (3.1). Given the Mach number of the undistrubed stream $M = q_\infty/c_\infty$, we can find q_∞ and hence Q_∞ by the use of numerical tables. Hence Q is known for each value of θ_N. Then, by using the tables, q and $K^{\frac{1}{2}}Qq^{-1}$ are found. Now, from (2.9) and (3.2) we have

$$x = 0, \qquad y = \int_0^\eta \frac{Q}{q}K^{\frac{1}{2}}\,d\eta = \int_{\pi/2}^{\theta} \frac{Q}{q}K^{\frac{1}{2}}\frac{d\eta}{d\theta}\,d\theta. \tag{3.3}$$

The integration can be carried out most conveniently by Simpson's rule, the ordinate y of the flat plate in the physical z-plane being thus found for each θ_N. In particular, the width l of the plate is given by

$$l = 2y(0) = -2\int_0^{\pi/2} (Q/q)K^{\frac{1}{2}}(d\eta/d\theta)\,d\theta. \tag{3.4}$$

Next, remembering that the pressure acting on the rear surface of the plate is everywhere equal to the pressure p_∞ at infinity, we have for the drag of the plate

$$X = 2\int_0^{l/2} (p - p_\infty)\,dy, \tag{3.5}$$

where p is the pressure on the front surface of the plate. Hence the drag coefficient C_D is given by

$$C_D = \frac{X}{\frac{1}{2}\rho_\infty q_\infty^2 l} = -\frac{4}{\gamma M^2}\frac{p_0}{p_\infty}\frac{l_0}{l}\frac{1}{l_0}\int_0^{\pi/2}\left(\frac{p}{p_0}-\frac{p_\infty}{p_0}\right)\frac{Q}{q}K^{\frac{1}{2}}\frac{d\eta}{d\theta}\,d\theta. \tag{3.6}$$

Here p/p_0, p_∞/p_0, M are found from the tables, l/l_0 given by (3.4), and the integration is performed quite similarly to (3.3). The results of calculation are given in Table 2.

In a previous paper [6] the senior author has obtained an analytical expression for C_D exact to M^6, which for $\gamma = 1.4$ becomes

$$2\pi/C_D = \pi + 4 - 2M^2 + 0.2898\,M^4 - 0.0028\,M^6 - 0.0044\,M^8. \tag{3.7}$$

In view of the smallness of the coefficients this formula is believed to be very accurate up to $M = 1$. Values of C_D/C_{D0} as calculated by the use of (3.7) are also included in Table 2, where the column $0(M^2)$ gives the values calculated by neglecting $0(M^4)$ and the column $0(M^6)$ those calculated by employing the full

TABLE 2

M	l/l_0	C_D	C_D/C_{D0}			
			Present method	Demtchenko	$O(M^4)$	$O(M^6)$
0	1.000	0.880	1.000	1.000	1.000	1.000
0.3	0.997	0.899	1.025	1.025	1.026	1.026
0.6	0.991	0.994	1.106	1.098	1.112	1.106
0.7	0.988	0.997	1.147	1.132	1.159	1.146
0.8	0.985	1.037	1.196	1.170	1.218	1.194
0.9	0.983	1.085	1.255	1.213	1.293	1.251
1.0	0.982	1.161	1.344	1.258	1.389	1.317

expression. Further, values obtained by the use of Demtchenko's approximate formula [1]:

$$2\pi/C_D = \pi + 4(\rho_\infty/\rho_0) \tag{3.8}$$

are also given for comparison. Curves of C_D/C_{D0} vs. M as calculated by the various methods are shown in Fig. 1. It will be seen that the results obtained by the present method of approximation are very accurate even near the extreme case $M = 1$.

4. Circular cylinder. Among the cases of subsonic flow past obstacles hitherto investigated the most reliable one is certainly that of flow past a circular cylinder. In fact, various formulae concerning the flow have been obtained by the senior author [2] exactly to M^6, and also T. Simazaki, in an unpublished paper, obtained the formulae correct to M^8. Therefore this case may be expected to be most suitable for testing the accuracy of any approximate method proposed. According to a previous investigation as yet unpublished, the critical Mach

number, at which the fluid velocity is first attained by the local velocity of sound, seems to be about 0.39 for the case of a circular cylinder. Therefore it will be natural to expect that the convergence of the series in M should probably be good for $M = 0.35$. Thus we have considered the velocity distribution on the surface of a circular cylinder at $M = 0.35$.

We have clearly in this case

$$\zeta_P = e^{i\theta}, \qquad Q = 2Q_\infty \sin\theta, \tag{4.1}$$

the radius of the cylinder being taken as unity. Hence, by (2.1)

$$x = \int_\theta^{\pi/2} (Q/q)K^{\frac{1}{2}} \sin\theta \, d\theta, \qquad y = \int_0^\theta (Q/q)K^{\frac{1}{2}} \cos\theta \, d\theta. \tag{4.2}$$

The procedure of calculation is quite similar to the preceding case. Again, it is sufficient to consider only 11 values of θ: $\theta_N = N\pi/20$ ($N = 0, 1, \cdots, 10$). It

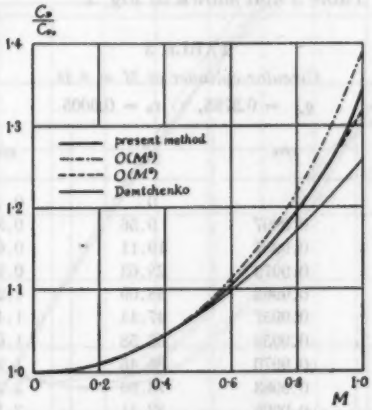

FIG. 1. Variation of the drag coefficient of a flat plate with the Mach number.

turns out that the circle in the ζ-plane is transformed to a profile P in the physical z-plane which is very similar to an ellipse of thickness ratio $t = 1.1208$.

It may be natural to expect that a nearly circular cylinder in the z-plane would be obtained from a certain ellipse in the ζ-plane. Accordingly we take an ellipse of thickness ratio t in the ζ-plane. Now we have

$$\zeta_P = e^{i\theta} + \sigma^2 e^{-i\theta}, \qquad \sigma^2 = (1 - t)/(1 + t), \tag{4.3}$$

$$\frac{Q}{Q_\infty} = \frac{2\sin\theta}{\sqrt{1 - 2\sigma^2\cos 2\theta + \sigma^4}}. \tag{4.4}$$

Hence, by (2.1), we get

$$x = (1 + \sigma^2)\int_\theta^{\pi/2} \frac{Q}{q} K^{\frac{1}{2}} \sin\theta \, d\theta, \qquad y = (1 - \sigma^2)\int_0^\theta \frac{Q}{q} K^{\frac{1}{2}} \cos\theta \, d\theta. \tag{4.5}$$

After some trials it was found that the value $t = 0.9165$ is suitable for the purpose. In Table 3 are given the polar coordinates (r, θ_0) of the profile P in the z-plane which corresponds to the ellipse of thickness ratio $t = 0.9165$ in the ζ-plane. Of course, we have

$$r = \sqrt{x^2 + y^2}, \qquad \theta_0 = \tan^{-1}(y/x). \tag{4.6}$$

It will be seen that the deviation of the profile from a circle is less than about 0.4 %. The magnitude of velocity q is given in Table 3 and graphically shown in Fig. 2. Since the velocity distribution over a circular cylinder in an incompressible fluid is clearly given by

$$(q/q_\infty) = 2 \sin \theta_0, \tag{4.7}$$

the effect of compressibility on the velocity is represented by

$$(\delta q/q_\infty) = (q/q_\infty) - 2 \sin \theta_0, \tag{4.8}$$

which is also given in Table 3 and shown in Fig. 2.

TABLE 3

Circular cylinder at $M = 0.35$

$q_\infty = 0.3788, \quad r_0 = 0.9005$

θ (deg.)	r	r/r_0	θ (deg.)	q/q_∞	$\delta q/q_\infty$
0	0.9005	1	0	0	0
9	0.9002	0.9997	9.56	0.3173	−0.0149
18	0.8993	0.9987	19.11	0.6283	−0.0265
27	0.8981	0.9973	28.63	0.9272	−0.0311
36	0.8972	0.9963	38.09	1.2083	−0.0255
45	0.8966	0.9957	47.44	1.4673	−0.0058
54	0.8968	0.9959	56.58	1.6994	0.0301
63	0.8978	0.9970	65.45	1.9000	0.0808
72	0.8990	0.9983	73.96	2.0602	0.1381
81	0.9001	0.9996	82.11	2.1674	0.1863
90	0.9005	1	90.00	2.2049	0.2049

In the above-mentioned paper [2], the senior author has given the expression for the velocity on the surface of a circular cylinder which is correct to M^6. Employing this formula, q/q_∞ as well as $\delta q/q_\infty$ were calculated for $M = 0.35$. The results of calculation are shown in Fig. 3 in order to compare with those obtained by the present method. Curves marked by M^2, M^4, M^6 represent respectively the results obtained by taking terms up to and including M^2, M^4, M^6 in the above-mentioned formula. It will be seen that the present method gives very good approximation to the true solution to be expected with sufficient accuracy from the successive approximations of the method of M^2-expansion. The departure which is observed in the region $40° < \theta < 70°$ is probably to be accounted for by considering the slight deformation of the profile from a true circle.

Lastly, the dotted line in Fig. 2 gives the velocity distribution as calculated

by a simplified procedure, which consists in entirely neglecting the deformation of the profile. Thus the value of q corresponding to that of Q as given by (4.1) is simply taken to be the velocity of the compressible fluid flow over the circular cylinder. It will be observed that such a simple procedure, which was shown in a previous paper [8] to give good approximation for thin profiles, will overestimate the effect of compressibility on the velocity distribution in the case of thick profiles.

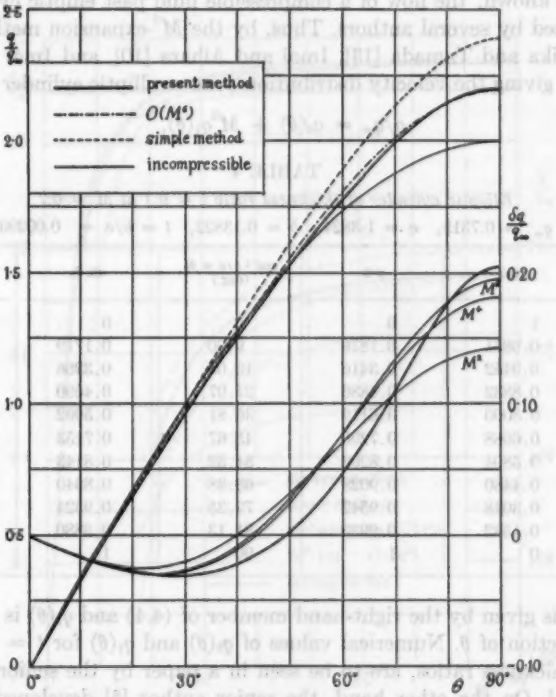

FIG. 2. Velocity distribution over a circular cylinder at $M = 0.35$.

5. Elliptic cylinder. As the last example, an elliptic cylinder of thickness ratio 1/10 at $M = 0.7$ will be considered. The procedure of calculation is quite similar to that for a circular cylinder.

Taking $t = 0.1$ and making use of (4.3), (4.4), (4.5), we arrive at a nearly elliptic cylinder of thickness ratio $t = 0.1059$ for $M = 0.7$.

Next, after a few trials, it was found that an ellipse of thickness ratio $t = 0.0949$ in the ζ-plane corresponds to a profile of nearly elliptic form of thickness ratio $t = 0.1$. Coordinates of the profile as well as the velocity distribution round it are given in Table 4.

It will be seen that y/b is in fairly good agreement with $\sin\theta_0$, where

$$x = a\cos\theta_0 . \tag{5.1}$$

This shows clearly that deviation of the profile from a true ellipse is very slight. Indeed, the numerical value of the difference between y/b and $\sin\theta_0$, which may be regarded as a measure of the deviation of the profile from a true ellipse, must be reduced by a factor 10, in order to consider the deviation in absolute magnitude.

As is well known, the flow of a compressible fluid past elliptic cylinders have been discussed by several authors. Thus, by the M^2-expansion method, Kaplan [12], Tomotika and Tamada [13], Imai and Aihara [10], and Imai [4] obtained the formula giving the velocity distribution over an elliptic cylinder in the form:

$$q/q_\infty = q_0(\theta) + M^2 q_1(\theta), \tag{5.2}$$

TABLE 4

Elliptic cylinder of thickness ratio $t = 0.1$ at $M = 0.7$

$q_\infty = 0.7318$, $a = 1.3824$, $b = 0.13822$, $t = b/a = 0.09999$

θ (deg.)	x/a	y/b	$\cos^{-1} x/a = \theta_0$ (deg.)	$\sin\theta_0$	q/q_∞
0	1	0	0	0	0
9	0.9851	0.1879	9.90	0.1719	0.9191
18	0.9452	0.3416	19.06	0.3266	1.0754
27	0.8832	0.4886	27.97	0.4690	1.1171
36	0.8006	0.6119	36.81	0.5992	1.1328
45	0.6988	0.7298	45.67	0.7153	1.1405
54	0.5804	0.8209	54.52	0.8143	1.1446
63	0.4480	0.9028	63.38	0.8940	1.1468
72	0.3048	0.9542	72.25	0.9524	1.1481
81	0.1542	0.9933	81.13	0.9880	1.1490
90	0	1	90	1	1.1492

where $q_0(\theta)$ is given by the right-hand member of (4.4) and $q_1(\theta)$ is a very complicated function of θ. Numerical values of $q_0(\theta)$ and $q_1(\theta)$ for $t = 1, 0.5, 0.1$, t being the thickness ratios, are to be seen in a paper by the senior author and Kawada [11]. On the other hand, the senior author [5] developed a so-called method of thin-wing-expansion which is essentially an extension of the linear theory of Prandtl and Glauert, and as an application obtained a formula for the velocity distribution over an elliptic cylinder in the form [3]:

$$\frac{q}{q_\infty} = 1 + \frac{t}{\sqrt{1-M^2}} - \frac{t^2}{2}\left\{\cot^2\theta + \frac{M^2}{1-M^2}\left(1 + \frac{\gamma+1}{4}\frac{M^2}{1-M^2}\right)\frac{\cos 2\theta}{\sin^2\theta}\right\}. \tag{5.3}$$

Values of q/q_∞ calculated by (5.2) and (5.3) as well as by the present method are shown in Fig. 3. Values of q/q_∞ obtained by the use of the simplified theory

neglecting the deformation of the profile are also shown in Fig. 3. Contrary to the previous case of a circular cylinder, agreement of the results obtained by the simple and the more elaborate method is now fairly good, as might be expected from the small thickness of the elliptic cylinder under consideration.

At such a high Mach number as $M = 0.7$ the results obtained by (5.2) would naturally be unreliable, since quantities of the order M^4 and higher powers are neglected. However, in the region near the stagnation point, the formula would probably give a good approximation. On the other hand, (5.3) should furnish an

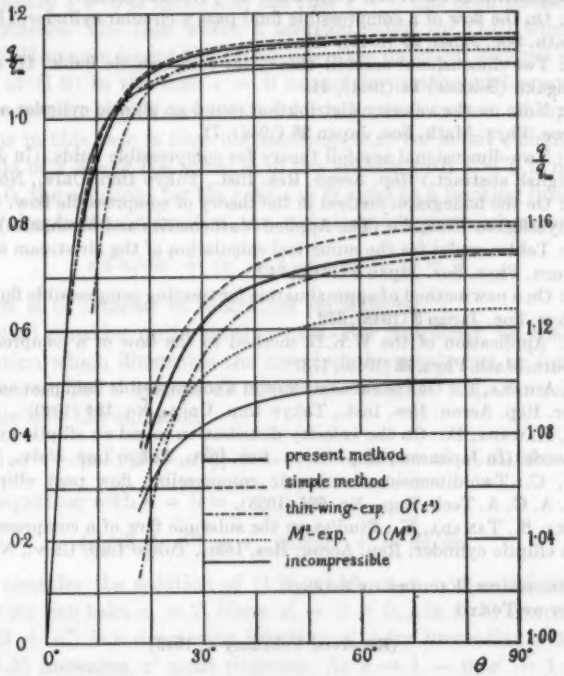

FIG. 3. Velocity distribution over an elliptic cylinder of thickness ratio $t = 0.1$ at $M = 0.7$.

excellent approximation in view of the small thickness of the profile, though it would greatly be in error near the stagnation point.

The agreement of the result obtained by the present method with that by (5.2) near the stagnation point and with that by (5.3) in the region of maximum velocity may be regarded as showing the accuracy of the present method.

6. Summary. A new method of approximation based on the W.K.B. method for treating the subsonic flow of a compressible fluid past a profile which was

proposed in a previous paper [9] is applied to three typical cases of flow, namely the flow past a flat plate accompanied with a dead air region, the flow past a circular cylinder at $M = 0.35$, and the flow past an elliptic cylinder of thickness ratio $t = 0.1$ at $M = 0.7$, M being the Mach number of the undisturbed flow. It is confirmed that the method furnishes very good approximation, although the amount of the necessary computational work is comparatively small.

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(Received February 4, 1949)

AN ORDINARY DIFFERENTIAL EQUATION WITH AN INTERVAL OF STABILITY, A SEPARATION POINT, AND AN INTERVAL OF INSTABILITY

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The relationship between the solutions of

$$\epsilon\, d^2u/dx^2 + f(x, u, du/dx, \epsilon) = 0. \tag{1.0}$$

for the case where $\epsilon \neq 0$ is small and the case $\epsilon = 0$ has been considered under various hypotheses. The case where a solution exists for (1.0) when $\epsilon = 0$ is treated in [2]. However a problem of considerable practical interest arises when the solution of (1.0) in the case $\epsilon = 0$ cannot be continued beyond a certain point. Under certain hypotheses, stated for a general system of equations [1], what happens in this case is that the solution, $u(x)$ for small ϵ changes rapidly near the value of x where the solution of the system with $\epsilon = 0$ is not prolongable. As $\epsilon \to 0$, the solutions for the case $\epsilon \neq 0$, tend to a limiting function which actually has simple discontinuities. An equation where this happens is

$$\epsilon\, d^2u/dx^2 + (u^2 - 1)\, du/dx + u = 0. \tag{1.1}$$

This equation is of interest in relaxation oscillations.

Here we shall consider quite another case. For simplicity we shall discuss a specific equation which illustrates the case but the result can be formulated as a general theorem for a system of equations. This we shall do subsequently.

We consider now the equation

$$\epsilon u'' - [4u'/(3 + u'^4)] + u = 0 \quad \text{where} \quad u' = du/dx. \tag{1.2}$$

The related equation with $\epsilon = 0$ is

$$4z'/(3 + z'^4) = z \quad \text{where} \quad z' = dz/dx. \tag{1.3}$$

Suppose we consider the solution of (1.3) which at $x = 0$ satisfies $z_0 = 8/19$. Then clearly we can take $z_0' = 2$. Since $z_0' = 2 > 0$, z in (1.3) increases with x. Clearly $4a/(3 + a^4)$ is a decreasing function of a for increasing $a > 1$ so that when z in (1.3) increases, z' must decrease. As $z \to 1 - 0$, $z' \to 1 + 0$. Since $dx = dz/z'$ this happens for a finite value of x which we can denote by x_1. Clearly

$$x_1 = \int dx = \int dz/z' < \int_{8/19} dz/1 = \tfrac{11}{19}.$$

This solution $z(x)$ of (1.3) cannot be prolonged beyond $x = x_1$. Indeed since $z_1' = 1 > 0$, if $z(x)$ were prolonged, z would have to increase beyond $z = 1$. However since $4a/(3 + a^4) \leqq 1$ for all real a this is impossible.

For small $\epsilon > 0$ we shall see that the solution of (1.2) $u(x)$ starting at $x = 0$

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with $u_0 = 8/19$ and $u_0' = 2$ will remain close to $z(x)$ until $x \to x_1 - 0$. (Actually it suffices to take u_0' in some such range as $1 \leqq u_0' \leqq \epsilon^{-9/10}$ but for simplicity we choose the initial values of u and u' identical with those of z and z'.) Obviously (1.1) is continuable from any finite values of u and u'. What happens to $u(x)$ beyond the point $x = x_1$, is that $u(x)$ goes into rapid oscillation of amplitude approximately 1 and period $2\pi\epsilon^{\frac{1}{2}}$. The graph of $u(x)$ is shown in Fig. 1 where the broken curve represents $z(x)$ for $x < x_1$. As $\epsilon \to +0$, $u(x) \to z(x)$, $0 \leqq x < x_1$, while for $x > x_1$, $u(x)$ oscillates as indicated above and clearly approaches no limit since its period tends to zero as $\epsilon \to +0$ while its amplitude tends to 1.

The behavior is highly interesting from the viewpoint of analogy with a problem in fluid mechanics. For $u(x)$ we have highly stable behavior in the interval $0 \leqq x \leqq x_1$. At x_1, or more precisely in an interval which shrinks to the point x_1 as $\epsilon \to 0$, a sudden change occurs. For $x > x_1$, we have a situation of rapid

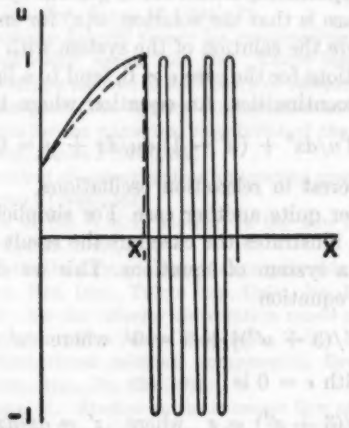

FIG. 1

oscillation which we can describe as unstable since small changes in parameters affect the period and phase in this oscillatory interval very much and small changes in the boundary conditions affect the phase. The point x_1 can be regarded as a separation point.

In fluid mechanics the physical situation in viscous incompressible flow in two dimensions is presumably described by the equation

$$\phi_x \Delta\phi_y - \phi_y \Delta\phi_x = \epsilon \Delta\Delta\phi \tag{1.4}$$

where Δ is the Laplace operator. Experiment for cases where ϵ is small suggests that in the case of flow around an obstacle there is a stable region separated from an unstable region in the (x, y) plane. Because of the difficulty of dealing with (1.4) mathematically, I considered analogous problems for simpler partial differential equations and was led to the conclusion that the occurrence of a

stable zone adjoining an unstable one can arise not only for a partial differential equation but even for an ordinary differential equation. Thus the much greater topological possibilities available to a function of two independent variables (as contrasted with a function of one independent variable) is not required for the general situation characterized by a stable zone adjoining an unstable zone as $\epsilon \to 0$. The equation (1.2) is such an example. It should be observed that the equation (1.4) is extremely difficult to analyze mathematically and that the analogy with (1.2) does not shed any new light on (1.4).

2. It remains now only to prove the statements we have made concerning the behavior of the solution of (1.2) as $\epsilon \to +0$. We shall find it more convenient to replace ϵ by ϵ^2 getting instead of (1.2)

$$\epsilon^2 u'' - [4u'/(3 + u'^4)] + u = 0. \tag{2.0}$$

We shall study (1.2) in the Poincare (u, v) plane by regarding it as a system

$$\frac{du}{dx} = v, \qquad \epsilon^2 \frac{dv}{dx} = \frac{4v}{3 + v^4} - u. \tag{2.1}$$

On the curve

$$u = 4v/(3 + v^4) \tag{2.2}$$

we see that $dv/dx = 0$. Let P_0 be (8/19, 2) in the (u, v) plane. On the curve (2.2) in the first quadrant of the (u, v) plane the solutions of (2.1) have $du/dx > 0$, $dv/dx = 0$.

Thus the arc of the curve (2.2), P_0E0 in Fig. 2 is cut by the solutions of (2.1) from left to right. Moreover to the right of P_0E0, $dv/dx < 0$. Thus the solution of (2.1) starting at P_0 descends with v decreasing and u increasing until the point P_2 on $v = 0$ is reached. Clearly the solution of (2.1), $P_0P_1P_2$ lies to the right of P_0EF. We shall now show that when $\epsilon \to 0$, $P_0P_1P_2 \to P_0EF$. To do this we construct the curve $ABCD$ as follows. AB is an arc of the curve

$$u = 4(v - \epsilon)/[3 + (v - \epsilon)^4] \tag{2.3}$$

where at A, $u_A = 8/19$ and $v_A = 2 + \epsilon$. At B, $u_B = 1 - \epsilon$. From this and (2.3) we find easily that at B, $1 + \frac{1}{2}\epsilon^{1/2} < v_B < 1 + \epsilon^{1/2}$. BC is a horizontal line with the u coordinate of C, $u_C = 1$. CD is a line with slope $-\epsilon^{-1/2}$. Clearly at D, $u_D < 1 + \epsilon^{1/2} + \epsilon$. In the first place it is clear that as $\epsilon \to 0$, $ABCD \to P_0EF$. In the second place it is easy to verify that the solutions of (2.1) cut $ABCD$ from above to below. This is self-evident on BC. On AB we have for the solutions of (2.1)

$$\frac{dv}{du} = \frac{1}{\epsilon^2 v}\left(\frac{4v}{3 + v^4} - u\right).$$

Replacing u above by its value in (2.3) we find since $v_B > 1 + \frac{1}{2}\epsilon^{1/2}$,

$$dv/du < -\tfrac{1}{4}\epsilon^{-1/2}$$

for small ϵ. On the other hand differentiating (2.3) we find that in the range of v in question the slope of (2.3) satisfies

$$dv/du > -\tfrac{1}{2}\epsilon^{-1/2}$$

for small ϵ. Clearly then for small ϵ the solutions of (2.1) cut AB from above to below. On CD we find similarly that the slope of the solutions of (2.1) satisfy

$$dv/du < -1/10\epsilon$$

For small ϵ the above is certainly $< -\epsilon^{-1/2}$. Thus $P_0P_1P_2$ cannot intersect $ABCD$ and therefore must lie below $ABCD$.

In the (x, u) plane shown in Fig 1 we have from (2.1) $dx = du/v$. As $\epsilon \to 0$, $P_0P_1 \to P_0E$ which is given by (2.2). The relation $dx = dz/z'$ and a comparison of (2.2) with (1.3) shows that in the (x, u) plane $u(x) \to z(x)$ as $\epsilon \to 0$ for $0 \leqq x < x_1$.

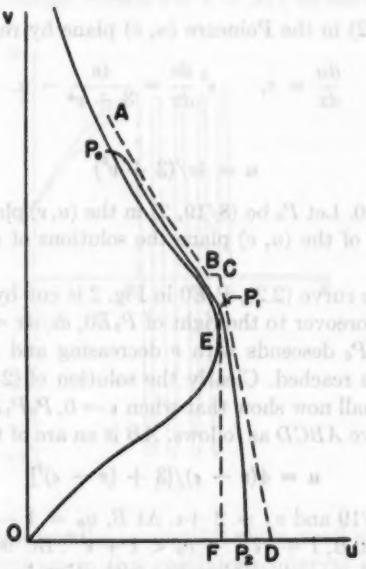

FIG. 2

In proceeding beyond P_1 we introduce the variables r and θ given by

$$u = r\cos\theta, v = (1/\epsilon)\, r\sin\theta.$$

Clearly $r^2 = u^2 + \epsilon^2v^2$, $\tan\theta = \epsilon v/u$. From these and (2.1) we have

$$\frac{dr^2}{dx} = \frac{8v^2}{3+v^4} \geqq 0, \qquad \frac{d\theta}{dx} = \frac{1}{\epsilon}\left(-1 + \frac{\cos\theta}{r}\,\frac{4v}{3+v^4}\right). \tag{2.4}$$

Since $|\, 4v/(3+v^4)\,| \leqq 1$, we see that if $r > 1$, $d\theta/dx < 0$. Thus for $r > 1$ the point (u, v) moves clockwise in the (u, v) plane with r increasing. We shall appraise the change in r when (u, v) traverses a quadrant of the (u, v) plane.

It is convenient to make the change of variables $u = y + 1$ and $v = w + 1$ in (2.1) getting

$$\frac{dy}{dx} = w + 1, \qquad \epsilon^2 \frac{dw}{dx} = -\frac{w^2 + 4w + 6}{3 + (w+1)^4} w^2 - y. \tag{2.5}$$

Now let P_1 represent the same point as shown in Fig. 2. It is the point where the trajectory (u, v) first reaches $u = 1$. Its (u, v) coordinates, (u_1, v_1) are $(1, 1 + w_1)$. Clearly from the earlier consideration $0 < w_1 < \epsilon^{1/2}$. We shall assume here that $w_1 > \epsilon^{2/3}$. The points where the descending v coordinate first becomes $1 + \epsilon^{2/3}$, $1 - \epsilon^{2/3}$ and 0 respectively we shall denote by P_2, P_3 and P_4. (Note that P_2 here is not P_2 of Fig. 2.) Since $u' = v > 0$, $1 = u_1 < u_2 < u_3 < u_4$. We now proceed from P_1 to P_2. Since $u > 1$, $y > 0$ and since $1 > w > w_2$ we have from (2.5)

$$-\epsilon^2 \frac{dw}{dx} \geqq \frac{w^2 + 4w + 6}{3 + (w+1)^4} w^2 \geqq \tfrac{1}{2} w^2.$$

Or

$$x_2 - x_1 \leqq -2\epsilon^2 \int_1^{\epsilon^{2/3}} dw/w^2 < 2\epsilon^{4/3}. \tag{2.6}$$

From P_2 to P_3 we have also from (2 5)

$$-\epsilon^2 \, dw/dx \geqq y = y_2 + \int_{x_2}^{x} (1 + w)\, dx. \tag{2.7}$$

Since $|w| \leqq \epsilon^{2/3}$ and $y_2 > 0$ we have

$$-\epsilon^2 \, dw/dx > \frac{1}{2} \int_{x_2}^{x} dx = \tfrac{1}{2}(x - x_2).$$

Thus integrating we get

$$2\epsilon^2 \epsilon^{2/3} > \tfrac{1}{4}(x_3 - x_2)^2$$

or

$$x_3 - x_2 < 3\epsilon^{4/3}. \tag{2.8}$$

From P_3 to P_4 we proceed just as from P_1 to P_2 and find $x_4 - x_3 < 2\epsilon^{4/3}$. With (2.6) and (2.8) we have

$$x_4 - x_1 < 7\epsilon^{4/3}. \tag{2.9}$$

From (2.9) we see that in the (x, u) plane $x_4 \to x_1$, as $\epsilon \to 0$.

We have assumed that $w_1 > \epsilon^{2/3}$. In any case $w_1 > 0$ and if $w_1 < \epsilon^{2/3}$ we can eliminate the first step of our argument (from P_1 to P_2) and make an obvious modification in the second part with the result that (2.9) still holds.

We also have from (2.5) in the range $0 \geqq w \geqq -\epsilon^{2/3}$, denoting y at $w = 0$ by y_0 and x by x_0

$$-\epsilon^2 \, dw/dx < 2w^2 + y_0 + \int_{x_0}^{x} (w + 1)\, dx < 2\epsilon^{4/3} + y_0 + (x - x_0).$$

If $y_0 \leqq \epsilon^{4/3}$ we have on integrating

$$\epsilon^{8/3} < 3\epsilon^{4/3}(x_3 - x_0) + \tfrac{1}{2}(x_3 - x_0)^2.$$

From this $x_3 - x_0 > \frac{1}{4}\epsilon^{4/3}$ from which

$$x_3 - x_1 > \tfrac{1}{4}\epsilon^{4/3}.$$

Since from (2.5), $dy = (w + 1)\,dx$ we have using the above

$$y_3 - y_1 \geqq \tfrac{1}{2}(x_3 - x_1) > \tfrac{1}{8}\epsilon^{4/3} \tag{2.10}$$

if $y_0 < \epsilon^{4/3}$. If $y_0 > \epsilon^{4/3}$, since $y_1 = 0$, we certainly have (2.10). Thus (2.10) holds in every case. From (2.10) we see that $u_3 = 1 + y_3 > 1 + \frac{1}{8}\epsilon^{4/3}$. Since r is increasing we see that beyond P_3, $r > 1 + \frac{1}{8}\epsilon^{4/3}$.

After P_4, $P(u, v)$ moves clockwise in the plane with r increasing. We appraise the increase in r in a quadrant. This appraisal is more difficult in the first and third quadrants than in the second and fourth. Let us examine the situation in the first quadrant. Let us denote the point in the first quadrant where $u = 0$ by $P_5(u_5, v_5)$ where $r = r_5$. Let us denote the successive ponts where $v = \epsilon^{-2/3}$ by $P_6(u_6, v_6)$, where $v = 2$ by $P_7(u_7, v_7)$, and where $v = 0$ by $P_8(u_8, v_8)$. From P_5 to P_6 we have

$$dr^2 = \frac{8v^2}{3 + v^4}\,dx = \frac{8v}{3 + v^4}\,du.$$

Thus since $v \geqq \epsilon^{-2/3}$, $r_6^2 - r_5^2 \leqq 8\epsilon^2 r_6$. Or

$$r_6 - r_5 \leqq 8\epsilon^2.$$

From P_6 to P_7, since $u^2 + \epsilon^2 v^2 = r^2 > 1$ and $\epsilon^2 v^2 \leqq \epsilon^{2/3}$, $u^2 > 1 - \epsilon^{2/3} > \frac{3}{4}$. We have using (2.1) and (2.4)

$$\begin{aligned} dr^2 = \frac{8v^2}{3 + v^4}\,\frac{\epsilon^2\,dv}{[4v/(3 + v^4)] - u} &< -\frac{8\epsilon^2 v^2\,dv}{3 + v^4}\,\frac{1}{\frac{3}{4} - [4v/(3 + v^4)]} \\ &< -32\epsilon^2\,\frac{v^2\,dv}{3 + v^4}. \end{aligned} \tag{2.11}$$

Thus

$$r_7^2 - r_6^2 < 32\epsilon^2 \int_2^\infty [v^2\,dv/(3 + v^4)] < 16\epsilon^2$$

or $r_7 - r_6 < 16\epsilon^2$.

At P_7 we have, since $r^2 > 1 + \frac{1}{4}\epsilon^{4/3}$ as remarked after (2.10),

$$u_7^2 + 4\epsilon^2 = r_7^2 > 1 + \tfrac{1}{4}\epsilon^{4/3}.$$

Thus $u_7 > 1$. We can therefore proceed from P_7 to P_8 by exactly the same argument as yielded (2.9). We have therefore $x_8 - x_7 < 7\epsilon^{4/3}$. Using

$$dr^2 = [8v^2/(3 + v^4)]\,dx < 12\,dx$$

we have $r_8^2 - r_7^2 < 12(x_8 - x_7) < 84\epsilon^{4/3}$ or $r_8 - r_7 < 84\epsilon^{4/3}$. Combining our results we have $r_8 - r_6 < 85\epsilon^{4/3}$ for small ϵ.

In the second and fourth quadrants the result is easier to get since in (2.11) the terms $-u$ and $4v/(3 + v^4)$ have the same sign thereby allowing us to go from $v = 0$ to $v = -\epsilon^{-2/3}$ in one step. In any case in each quadrant r grows less than $85\epsilon^{4/3}$.

We now consider the change in x when $P(u, v)$ traverses the fourth (or second) quadrant. P goes from $P_3(u_3, 0)$ to $P_9(0, v_9)$ where $r_3 = u_3$ and $r_9 = -\epsilon v_9$. We have since $dx = du/v$,

$$\int dx = -\epsilon \int_{u_3}^{0} \frac{du}{(r^2 - u^2)^{1/2}}.$$

Since r is increasing

$$\int dx < \epsilon \int_0^{u_3} \frac{du}{(u_3^2 - u^2)^{1/2}} = \frac{\pi}{2}\epsilon.$$

Similarly

$$\int dx > \epsilon \int_0^{u_3} \frac{du}{(u_9^2 - u^2)^{1/2}} = \epsilon \sin^{-1} \frac{r_3}{r_9}.$$

Since $r_9 - r_3 < 85\epsilon^{4/3}$, we have

$$\int dx > \tfrac{1}{2}\pi\epsilon - 20\epsilon^{5/3}$$

Thus in any case the change in x in the second or fourth quadrant tends to $\frac{1}{2}\pi\epsilon$ as $\epsilon \to 0$.

In the same way in the first or third quadrant we find

$$\int dx \geqq \tfrac{1}{2}\pi\epsilon.$$

But the upper bound is more complicated here. We use the notation $P_5(0, v_5)$ and $P_3(u_3, 0)$ for the end points of an arc in the first quadrant. We denote the point where $u = \epsilon v_5 = r_5$ by $P_{10}(u_{10}, v_{10})$. Since r is increasing

$$\int_{x_3}^{x_{10}} dx \leqq \epsilon \int_0^{r_5} \frac{du}{(r_5^2 - u^2)^{1/2}}$$

Thus $x_{10} - x_5 \leqq \frac{1}{2}\pi\epsilon$. Now since $r_{10}^2 = r_5^2 + \epsilon^2 v_{10}^2$ and since $r_{10} - r_5 < 85\epsilon^{4/3}$ we have $v_{10} < 20 r_5^{1/2} \epsilon^{-1/3}$. For $2 < v < v_{10}$ we have

$$dx = -\frac{\epsilon^2\, dv}{u - [4v/(3 + v^4)]} < -\frac{\epsilon^2\, dv}{1 - \frac{1}{2}} = -2\epsilon^2\, dv.$$

Thus in this range

$$\int dx < 40 r_5^{1/2}\, \epsilon^{5/3}.$$

For $0 < v < 2$ we have aready seen that $\int dx < 7\epsilon^{4/3}$. Thus we have

$$x_3 - x_b \leqq \tfrac{1}{2}\pi\epsilon + 7\epsilon^{4/3} + 40r_b^{1/2}\epsilon^{5/3}.$$

This shows that in the first and third quadrants the change in x tends to $\frac{1}{2}\pi\epsilon$ as $\epsilon \to 0$. Beyond the point x_1 in the graph of $u(x)$ in the (x, u) plane we see that as $\epsilon \to 0$ the solution oscillates with a period tending to $2\pi\epsilon$. Moreover in any finite interval of x the number of oscillations is $0(1/\epsilon)$. Since the growth in r per oscillation is $0(\epsilon^{4/3})$, we see that the growth in r over any finite iterval is $0(\epsilon^{1/3})$. Thus as $\epsilon \to 0$ the amplitude of the oscillations over any finite interval in x tends to 1. This completes the proof.

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(Received December 8, 1948)

ON CERTAIN INTEGRAL EQUATIONS IN DIFFRACTION THEORY

By John W. Miles

The two dimensional diffraction problems of a plane wave incident on an infinite slit or ribbon were solved (in closed form) by Sieger,[1] following a suggestion by Wien[2] and an earlier discussion by Schwarzchild.[3] Numerical results were given by Morse.[4] The integral equation formulation of diffraction problems has received attention in more recent papers.[5, 6, 7, 8]

If the (y) component of the magnetic field, assumed parallel to a slit bounded by $x = \pm 1$ in the plane $z = 0$, is taken as the stream function $\psi(x, z)$, and the incident wave nomal makes an angle θ with the screen $(z = 0)$, viz.

$$\psi_i(x, z, t) = \exp\,[ik(x \cos\theta + z \sin\theta - ct)] \tag{1}$$

where k is the wave number, and the time variation is harmonic (angular frequency $kc = \omega$), the fields may be determined from the strength of the tangential electric field in the slit, defined as

$$f(x) = -ik^{-1}\,(\partial\psi(x, z)/\partial z)_{z=0} \tag{2}$$

the time factor being suppressed. It is found that $f(x)$ must satisfy the integral equation

$$(k/2)\int_{-1}^{+1} d\xi\, H_0^{(1)}(k\,|\,x - \xi\,|)f(\xi) = \exp\,(ikx\cos\theta), \qquad |\,x\,| < 1 \tag{3a}$$

$$f(x) = 0, \qquad |\,x\,| > 1 \tag{3b}$$

Introducing the elliptic coordinates (u, v):

$$x = \cosh u \cos v, \quad y = \sinh u \sin v \tag{4}$$

the solution to the integral equation (3), in the notation of Morse, is found to be

$$f(x) = 4\pi k^{-1} \csc v \sum_{m=0}^{\infty} i^{m-1} e^{i\gamma_m} \lambda_m N_m^{-1} \sin \gamma_m S_{om}(k, \cos\theta)\, S_{om}(k, \cos v) \tag{5}$$

If the Fourier transform of $f(x)$ is introduced, Eqs. (3) become

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\lambda[1 - (\lambda/k)^2]^{-1/2} F(\lambda)\exp\,(i\lambda x) = \exp\,(ikx\cos\theta), \qquad |\,x\,| < 1 \tag{6a}$$

$$f(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} d\lambda F(\lambda)\exp\,(i\lambda x) = 0, \qquad |\,x\,| > 1 \tag{6b}$$

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The solution to Eqs. (6) is

$$F(\lambda) = 4\pi(ik)^{-1} \sum_{m=0}^{\infty} e^{i\gamma_m} \lambda_m N_m^{-1} \sin \gamma_m S_{e_m}(k, \cos\theta) R_{e_m}^1(k, \lambda/k) \tag{7}$$

If the conducting portions of the screen and the aperture are interchanged, so that the problem is to find the scattering due to a conducting ribbon bounded by $x = \pm 1$ in the plane $z = 0$, the total fields may be obtained by superimposing on the incident wave the radiation due to the surface current $h(x)$ flowing in the ribbon, where

$$h(x) = \tfrac{1}{2}[\psi(x, 0-) - \psi(x, 0+)] \tag{8}$$

The integral equation for $h(x)$ is found to be[9]

$$h(x) = 0, \qquad |x| > 1 \tag{9a}$$

$$\int_{-1}^{+1} d\xi\, |x - \xi|^{-1} H_1^{(1)}(k|x - \xi|)\, h(\xi) = 2 \sin\theta \exp(ikx\cos\theta), \qquad |x| < 1 \tag{9b}$$

and the solution is given by

$$h(x) = -4\pi \sum_{m=0}^{\infty} i^m (\lambda_m'/N_m') e^{i\gamma_m'} \sin \gamma_m' S_{0_m}(k, \cos\theta) S_{0_m}(k, x) \tag{10}$$

Similarly, the integral equation for the Fourier transform and its solution are given by

$$h(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\lambda\, H(\lambda) \exp(i\lambda x) = 0, \qquad |x| > 1 \tag{11a}$$

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} d\lambda\, [1 - (\lambda/k)^2]^{1/2} H(\lambda) \exp(i\lambda x) = \sin\theta \exp(ikx\cos\theta),$$

$$|x| < 1 \tag{11b}$$

$$H(\lambda) = 4\pi i (\lambda^2 - k^2)^{-1/2} \sum_{m=0}^{\infty} (\lambda_m'/N_m') e^{i\gamma_m'} \sin \gamma_m' S_{0_m}(k, \cos\theta) R_{0_m}^1(k, \lambda/k) \tag{12}$$

In using the foregoing results (and the following integrals) the contours must be indented under the branch point $\lambda = -k$ and over $\lambda = +k$, in order that the fields will satisfy appropriate boundary conditions at infinity.

While the foregoing results are of interest per se, they also lead to certain use-

[9] The singularity in the integrand of Eq. (9b) is due essentially with the assumption of an infinitely thin strip. This difficulty may be circumvented by replacing $|x - \xi|$ by $[(x - \xi)^2 + \epsilon^2]^{1/2}$, with the implication that the boundary conditions are applied at $z = \pm\epsilon$ and either integrating twice by parts before solving or taking the limit $\epsilon = 0$ after solving the equation as it stands. Similarly, the path of integration for Eq. (11) must be appropriately deformed in the complex λ plane to secure convergence of the integrals as stated, and Eq. (9b) may be interpreted as the Faltung inversion of Eq. (11b), assuming an appropriate path. It suffices to say here that the results stated in Eq. (13) et seq. obtain as proper limits of such a process.

ful integral relations among the Mathieu functions. Thus, it is found that

$$\int_{-\infty}^{\infty} dt(1-t^2)^{-1/2} R^1_{e_m}(k,t) \exp(ikxt) = i^m \lambda_m^{-1} \begin{cases} (1+i\cot\gamma_m)R^1_{e_m}(k,x), & |x|<1 \\ R^2_{e_m}(k,x), & |x|>1 \end{cases} \tag{13}$$

where the path of integration passes below and above the branch points $t = \mp 1$, respectively. That portion of the integral (13) from the path $t = -1$ to $t = +1$ is given by Stratton,[10] but the extension of the path appears to be new.

Considering the path for $|t| > 1$ and letting $t = \cosh\zeta$ yields

$$\int_0^{\infty} d\zeta R^1_{e_{2m}}(k,\cosh\zeta)\cos(kx\cosh\zeta) = \begin{cases} (-)^m(\pi/2)^{1/2}(\mu_{2m}/\lambda_{2m})S_{e_{2m}}(k,x), & |x|<1 \\ (-)^{m+1}(2\lambda_{2m})^{-1} R^2_{e_{2m}}(k,x), & |x|>1 \end{cases} \tag{14a}$$

$$\int_0^{\infty} d\zeta R^1_{e_{2m+1}}(k,\cosh\zeta)\sin(kx\cosh\zeta) = \begin{cases} (-)^m(\pi/2)^{1/2}(\mu_{2m+1}/\lambda_{2m+1})S_{e_{2m+1}}(k,x), & |x|<1 \\ (-)^{m+1}(2\lambda_{2m+1})^{-1} R^2_{e_{2m+1}}(k,x), & |x|>1 \end{cases} \tag{14b}$$

Another useful result is

$$\int_0^{2\pi} d\beta H_0^{(1)}(k\,|x-\cos\beta\,|)S^-_{e_m}(k,\cos\beta) = (8/\pi)^{1/2}(1+i\cot\gamma_m)\lambda_m^{-1} R^1_{e_m}(k,x), \quad |x|<1 \tag{15}$$

which is a continuation of the result given by Stratton[11] for $|x| > 1$.

Similar results for the odd functions are:

$$\int_{-\infty}^{\infty} dt R^1_{o_m}(k,t)\exp(ikxt) = (i^m/k\lambda'_m)\begin{cases} (\cot\gamma'_m - i)(1-x^2)^{-1/2} R^1_{o_m}(k,x), & |x|<1 \\ (x^2-1)^{-1/2} R^2_{o_m}(k,x), & |x|>1 \end{cases} \tag{16}$$

$$\int_0^{2\pi} d\beta H_1^{(1)}(k\,|x-\cos\beta\,|)S_{o_m}(k,\cos\beta)\sin\beta\,|x-\cos\beta\,|^{-1} = (8/\pi)^{1/2}(1+i\cot\gamma'_m)(ik\lambda'_m)^{-1}(1-x^2)^{-1/2} R^1_{o_m}(k,x), \quad |x|<1 \tag{17}$$

Additional integral relations may evidently be obtained by differentiating the foregoing results with respect to various parameters.

[10] J. A. Stratton, *Electromagnetic theory*, McGraw-Hill Co., New York (1941), p. 386, Eq. (87).

[11] ibid., p. 384, Eq. (70).

Similar results may be formulated for the problem of diffraction through a circular hole, leading to integral relations among the solutions to the Helmholtz equation in oblate spheroidal coordinates. An exact solution to the problem has been given by Bouwkamp,[12] and the integral equation formulation has been discussed by Sommerfeld,[6] by Levine and Schwinger,[8] and by the writer.[13]

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(Received December 13, 1948)

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RADIATION FROM A TRANSVERSE SLOT IN AN INFINITE CYLINDER*

By Charles H. Papas

Introduction. We consider the radiation from a transverse slot cut in an infinitely long, perfectly conducting circular cylinder, Fig. 1. It is assumed that the slot is fed by a rectangular waveguide within the cylinder such that the electric field across the slot, i.e. across the mouth of the waveguide, has only an E_z-component. The source driving the waveguide within the cylinder establishes a dominant mode which travels towards the slot and is there partially reflected and partially transmitted into free space. Moreover, in the neighborhood of the slot, nonpropagating modes are generated. To solve the problem rigorously

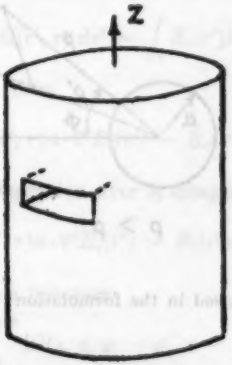

Fig. 1. Transverse Slot Cut on an Infinite Cylinder

would require the calculation of the radiated field from a knowledge of the dominant mode incident upon the slot. An analysis of this sort would be involved. The next best solution of the problem is to start from an assumed distribution of E_z across the slot. Then, by the well-known methods involving Green's identity and an appropiate Green's function the radiated field can easily be determined. The deviation of the values of the field thus obtained from the exact values depends upon the degree to which the assumed distribution across the surface of the slot approximates the actual distribution. In this note we use the Kirchhoff approximation, i.e., we assume that the distribution of E_z across the slot is the same as the distribution of the incident dominant mode. This approximation violates the boundary conditions at the slot to the extent of the higher nonpropagating modes and the reflected dominant mode.

* The research reported in this document was made possible through support extended Cruft Laboratory, Harvard University, jointly by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army and the U. S. Air Force, under ONR Contract N5ori-76, T. O. 1.

The radiation from a single transverse or longitudinal slot plays a fundamental role in the engineering design of directional arrays of slots just as the radiation pattern of a single cylindrical dipole antenna is a building block for directional dipole arrays. We do not discuss the longitudinal slot here, for the calculation of its radiation field follows very closely the procedure used in the case of the transverse slot.

Formulation of the Problem. At the surface of an infinitely long, perfectly conducting circular cylinder the tangential component of the electric field must disappear, except over that portion of the surface where a slot has been cut. Its values there depend upon the shape of the slot and the manner of excitation. If the slot is transverse and is fed from within the cylinder by a rectangular waveguide such that the incident dominant mode, i.e., the dominant mode traveling

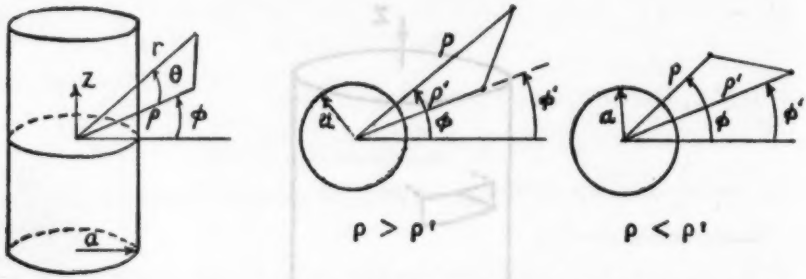

FIG. 2. Coordinate systems used in the formulation: Cylindrical coordinates ρ, ϕ, z; spherical coordinates r, θ, ϕ.

outward from the axis of the cylinder towards the mouth of the guide or slot, is $TE_{0,1}$, the only component which the E-field of this incident dominant mode possesses is E_z. In accordance with the Kirchhoff approximation we assume that the distribution of E_z across the slot equals the distribution of this dominant mode.

By confining our discussion to a transverse slot with a known distribution of E_z, we can recast the problem in terms of the classical Neumann problem of scalar fields: given the values of a scalar function $\psi(=E_z)$ on the surface enclosing a volume, find the value of ψ at each point within the volume. Considering the surface of the infinite cylinder and of the sphere at infinity as the enclosing surface, the value of ψ can be found at any point in the region surrounding the cylinder since ψ is known at every point on the bounding surface. Namely, $\psi = 0$ on the sphere at infinity and at all points on the surface of the cylinder except across the slot.

It is convenient to introduce the Green's function $G(r, r')$, which satisfies the inhomogeneous wave equation (1)

$$(\nabla^2 + k^2)G(r, r') = -\delta(r-r') \tag{1}$$

where $\delta(r-r')$ is the three-dimensional delta-function, r and r' are vectors extending from the origin to the observation and source points, respectively, $k = 2\pi/\lambda$, λ = free-space wavelength. Applying Green's theorem to $G(r, r')$ and $E_z(r')$ yields

$$\int_v [G(r', r)\,(\nabla'^2 + k^2)E_z(r') - E_z(r')\,(\nabla'^2 + k^2)G(r', r)]\,d\tau' = \int_{s_1+s_2} [G(r', r)n\cdot\nabla' E_z(r') - E_z(r')n\cdot\nabla' G(r', r)]\,ds' \tag{2}$$

where n is the outward normal unit vector; the surface integration is over s_1, the surface of the cylinder, and s_2, the surface of the sphere at infinity, and the volume integration is through the region enclosed by these surfaces. The first term of the volume integral disappears since $(\nabla^2 + k^2)E_z(r) = 0$ and in view of (1) the second term of the volume integral becomes

$$\int_v - E_z(r')(\nabla'^2 + k^2)G(r', r)\,d\tau' = \int_v E_z(r')\delta(r' - r)\,d\tau' = E_z(r)$$

Consequently, (2) takes the form

$$E_z(r) = \int_{s_1+s_2} [G(r', r)n\cdot\nabla' E_z(r') - E_z(r')n\cdot\nabla' G(r', r)]\,ds'$$

It can be shown that the integration over s_2 disappears, leaving

$$E_z(r) = \int_{s_1} [G(r', r)n\cdot\nabla' E_z(r') - E_z(r')n\cdot\nabla' G(r', r)]\,ds'.$$

In cylindrical coordinates this becomes

$$E_z(\rho, \phi, z) = -a\int_{-\infty}^{\infty}\int_0^{2\pi} [G(\rho, a, \phi - \phi', z - z')(\partial/\partial\rho')E_z(a, \phi', z') - E_z(a, \phi', z')(\partial/\partial\rho')G(\rho,a,\phi-\phi',z-z')]\,d\phi'\,dz'. \tag{3}$$

We are at liberty to assign homogeneous boundary conditions to $G(r', r)$ such that either $G(r', r)$ or $(\partial/\partial\rho')G(r', r)$ disappears on the surface of the cylinder. If we choose the former, i.e., $G(r, r') = 0$ for $\rho' = a$, (3) reduces to

$$E_z(\rho, \phi, z) = a\int_{-\infty}^{\infty}\int_0^{2\pi} E_z(a, \phi', z')(\partial\cdot\partial\rho')G(\rho, a, \phi - \phi', z - z')\,d\phi'\,dz'. \tag{4}$$

This is a useful form, since the term $E_z(a, \phi', z')$ is the distribution of E_z over the surface of the cylinder, it being zero at all points on the surface except across the slot.

The Green's Function. To find the explicit form of $G(r, r')$, we first write (1) in cylindrical coordinates.[1]

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right]G(\rho, \rho', \phi - \phi', z - z') = -\frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \phi')\delta(z - z'). \tag{5}$$

We choose the solution of (5) which satisfies the condition $G(\rho, a, \phi - \phi', z - z') = 0$. Let ζ be a complex variable. Multiplying (5) by $e^{-i\zeta z}$ and integrating over all values of z, we get

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2 - \zeta^2\right] G(\rho, \rho', \phi - \phi', \zeta) = -\frac{\delta(\rho - \rho')}{\rho}\,\delta(\phi - \phi') \tag{6}$$

where

$$G(\rho, \rho', \phi - \phi', \zeta) = \int_{-\infty}^{\infty} G(\rho, \rho', \phi - \phi', z)\, e^{-i\zeta z}\, dz.$$

We assume that

$$G(\rho, \rho', \phi - \phi', \zeta) = \frac{1}{2\pi}\sum_{m=-\infty}^{m=\infty} e^{im(\phi - \phi')}\, G_m(\rho, \rho', \zeta). \tag{7}$$

Substituting (7) into (6) and integrating with respect to ϕ from 0 to 2π, the m^{th} Fourier coefficient $G_m(\rho, \rho', \zeta)$ is seen to satisfy the equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + k^2 - \zeta^2 - \frac{m^2}{\rho^2}\right] G_m(\rho, \rho', \zeta) = \frac{-\delta(\rho - \rho')}{\rho}. \tag{8}$$

When $\rho \neq \rho'$ the right side of (8) disappears and we recognize it as being Bessel's equation whose solutions are combinations of the Bessel functions $J_m(x)$ and the Hankel functions $H_m^{(1)}(x)$, both of order m:

$$\begin{aligned} G_m(\rho, \rho', \zeta) = A_m H_m^{(1)}(\sqrt{k^2 - \zeta^2}\rho)[H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)J_m(\sqrt{k^2 - \zeta^2}\,\rho') \\ - H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho')J_m(\sqrt{k^2 - \zeta^2}\,a)] \end{aligned} \tag{9}$$

for $\rho > \rho'$, and

$$\begin{aligned} G_m(\rho, \rho', \zeta) = A_m H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho')[H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)J_m(\sqrt{k^2 - \zeta^2}\,\rho) \\ - H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)J_m(\sqrt{k^2 - \zeta^2}\,a)] \end{aligned} \tag{10}$$

for $\rho' > \rho$. Since $G_m(\rho, \rho', \zeta)$ must be continuous at $\rho = \rho'$, the factor A_m in (9) is identical to the A_m in (10). Multiplying (6) by ρ and integrating over a small interval which includes ρ', we find that

$$A_m = \frac{\pi i}{2}\,\frac{1}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)}. \tag{11}$$

Substituting (11) into (9), the explicit form of the m^{th} Fourier coefficient of the Fourier transform of the Green's function for $\rho > \rho'$ is expressed by

$$\begin{aligned} G_m(\rho, \rho', \zeta) = \frac{\pi i}{2}\,\frac{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)}\,[H_m^{(1)}\sqrt{k^2 - \zeta^2}\,a)J_m(\sqrt{k^2 - \delta^2}\,\rho') \\ - H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho')J_m(\sqrt{k^2 - \zeta^2}\,a)]. \end{aligned} \tag{12}$$

By applying the Fourier inversion theorem to (7), wherein the $G_m(\rho, \rho', \zeta)$ coefficients are given by (12), we obtain the following expression for the Green's function:

$$G(\rho, \rho', \phi - \phi', z - z') = \frac{i}{8\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{m=\infty} \frac{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)}$$
$$\times [H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a) J_m(\sqrt{k^2 - \zeta^2}\,\rho') - H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho') J_m(\sqrt{k^2 - \zeta^2}\,a)]$$
$$\times e^{im(\phi - \phi')} e^{i\zeta(z - z')}\, d\zeta \qquad (13)$$

where the integration with respect to the complex variable ζ is taken along the real axis of the ζ-plane with an upward indentation at the branch point $\zeta = -k$ and a downward one at the branch point $\zeta = +k$. Differentiating (13) with respect to ρ' and then setting $\rho' = a$, we obtain the form appropriate for (4). The expression connecting the values $E_z(\rho, \phi, z)$ with $E_z(a, \phi', z')$ is then found to be

$$E_z(\rho, \phi, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} E_z(a, \phi', z') \sum_{m=-\infty}^{m=\infty} \frac{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)} \qquad (14)$$
$$e^{im(\phi - \phi')} e^{i\zeta(z - z')}\, d\zeta\, d\phi'\, dz'.$$

In this expression $E_z(a, \phi', z')$ is the value of E_z on the surface of the cylinder and is therefore zero everywhere except across the slot, where it is set equal to the z-component of the E-field belonging to the incident dominant wave (Kirchhoff approximation). $E_z(\rho, \phi, z)$ is the z-component of the radiated E-field. The problem is to find $E_z(\rho, \phi, z)$ when $E_z(a, \phi', z')$ is given across the slot.

It is interesting to note that when $\rho = a$

$$E_z(a, \phi, z) = \int_{-\infty}^{\infty} \int_0^{2\pi} E_z(a, \phi', z') \delta(\phi - \phi) \delta(z - z')\, d\phi'\, dz',$$

since

$$\sum_{m=-\infty}^{m=\infty} e^{im(\phi - \phi')} = 2\pi\delta(\phi - \phi') \quad \text{and} \quad \int_{-\infty}^{\infty} e^{i\zeta(z - z')}\, d\zeta = 2\pi\delta(z - z').$$

Therefore $E_z(a, \phi, z) = E_z(a, \phi, z)$ when (a, ϕ, z) lies within the bounds of the slot and $E_z(a, \phi, z) = 0$ for all other points (a, ϕ, z).

Far Field. Our main task is to integrate (14). The exact evaluation for $E_z(\rho, \phi, z)$ in the neighborhood of the slot is difficult. However, by the method of steepest descents it is possible to arrive at an approximate value for points far away from the slot.

Commuting the order of summation and integration of (14) we get

$$E_z(\rho, \phi, z) = \frac{1}{4\pi^2} \sum_{m=0}^{\infty} \epsilon_m \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\phi' \cos m(\phi - \phi') E_z(a, \phi', z') \qquad (15).$$
$$\int_{-\infty}^{\infty} \frac{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)} e^{i\zeta(z - z')}\, d\zeta.$$

where $\epsilon_m = 1, 2$ for $m = 0$, $m \geq 1$, respectively, and integration with respect to ζ is along the real axis of the complex ζ-plane with an upward indentation at $\zeta = -k$ and a downward one at $\zeta = +k$.

We consider the integral [2]

$$\int_{-\infty}^{\infty} \frac{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2 - \zeta^2}\,a)}\, e^{i\zeta(z-z')}\, d\zeta. \tag{16}$$

The phase of $\sqrt{k^2 - \zeta^2}$ is specified along the real axis of the ζ-plane so that arg $\sqrt{k^2 - \zeta^2} = 0$ for $|\zeta| < k$ and arg $\sqrt{k^2 - \zeta^2} = \pi/2$ for $|\zeta| > k$. For large ρ and $\mu \neq 0$, where $\mu = \sqrt{k^2 - \zeta^2}$,

$$e^{i\zeta z} H_m^{(1)}(\mu\rho) \sim \left(\frac{2}{\pi\mu\rho}\right)^{1/2} e^{i(\mu\rho - (m\pi/2) - \pi/4)} e^{i\zeta z}. \tag{17}$$

The saddlepoint of $e^{i(\mu\rho + \zeta z)}$ is clearly $\zeta_s = kz/\sqrt{\rho^2 + z^2} = k \sin\theta$. When $\zeta = \zeta_s$, $e^{i(\mu\rho + \zeta z)} = e^{ik(\rho\cos\theta + z\sin\theta)}$ and in the neighborhood of ζ_s

$$\exp i(\mu\rho + \zeta_s) = \exp i\left[k(\rho\cos\theta + z\sin\theta) - \frac{\rho}{2k\cos^2\theta}(\zeta - \zeta_s)^2\right]. \tag{18}$$

The path of integration through the saddlepoint is so chosen that the imaginary part (I.P.) of the operand of exp in (18) remains constant as the path of integration through the saddle point is traversed. That is, we require

$$\text{I.P. } [ik(\rho\cos\theta + z\sin\theta)] = \text{I.P.}\left[ik(\rho\cos\theta + z\sin\theta) - \frac{i\rho}{2k\cos^2\theta}(\zeta - \zeta_s)^2\right]. \tag{19}$$

Now we let $\zeta - \zeta_s = Ae^{i\alpha}$ where α is the angle which the vector $\zeta - \zeta_s$ makes with the positive real axis of the complex ζ-plane. To satisfy (19) it is necessary that the real part of $A^2e^{2i\alpha}$ disappear. Hence, α can equal either $+\pi/4$ or $-\pi/4$. It is clear that $\alpha = \pi/4$ must be discarded; this leaves $\alpha = -\pi/4$ as the angle which the path must make.

We now transform from the ζ-plane to the τ-plane according to

$$\zeta = k\sin\tau. \tag{20}$$

where $\tau = \theta + i\psi$. The path of integration, Fig. 3, in the ζ-plane transforms

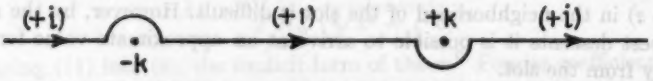

FIG. 3. Path of integration in ζ-plane

into the path C in the τ-plane, Fig. 4. C is unbent so that it starts at $-\pi/2 + i\,\infty$, passes through the saddlepoint $\theta = 0$, and then ends at $\pi/2 - i\,\infty$. A path passing through a saddlepoint $\theta = \theta_0$ would start at $\theta_0 - \pi/2 + i\,\infty$ and terminate

at $\theta_0 + \pi/2 - i\infty$, such as C_θ. In the τ-plane, (17) is expressed by

$$\left(\frac{2}{\pi\rho k \cos\tau}\right)^{\frac{1}{2}} e^{-i\frac{1}{2}(2m+1)\tau} e^{ikr\cos(\tau-\theta)} \tag{21}$$

Let $F(k\cos\tau)$ represent the factor $e^{-i\zeta z'}/H_m^{(1)}(\sqrt{k^2-\zeta^2}\,a)$ transformed to the τ-plane. With this and (21) it is clear that (16) is equal to

$$e^{-i\frac{1}{2}(2m+1)\tau}\int_{\theta_0-(\pi/2)+i\infty}^{\theta_0+(\pi/2)-i\infty}\left(\frac{2}{\pi\rho k\cos\tau}\right)^{\frac{1}{2}} e^{ikr\cos(\tau-\theta)}F(k\cos\tau)k\cos\tau\,d\tau, \tag{22}$$

This integration is carried out along C_θ, Fig. 4. Since r is large, (22) can be simplified by the following considerations. The chief contribution to (22) comes

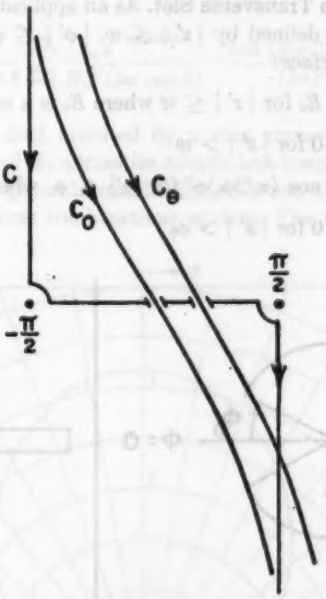

FIG. 4. Paths of integration in the τ-plane

from that portion of the path C_θ which lies in the immediate neighborhood of the saddlepoint, i.e. $\tau = \theta_0$, where the path makes an angle of $-\pi/4$ with the real axis. We can put $\tau - \theta = \eta e^{-i\pi/4}$ where η is real, and it follows that $\cos(\tau-\theta) \doteq 1 + i\eta^2/2$. With these approximations we have

$$\int_{-\infty}^{\infty}\frac{H_m^{(1)}(\sqrt{k^2-\zeta^2}\,\rho)}{H_m^{(1)}(\sqrt{k^2-\zeta^2}\,a)}e^{i\zeta(z-z')}\,d\zeta = 2e^{-i\frac{1}{2}(m+1)}\frac{e^{ikr}}{r}\frac{e^{-ikz'\sin\theta}}{H_m^{(1)}(ka\cos\theta)}. \tag{23}$$

It clearly follows from (23) and (15) that

$$E_z(\rho, \phi, z) = \frac{1}{2\pi^2} \sum_{m=0}^{\infty} \frac{\epsilon_m e^{-i(m+1)\pi}}{H_m^{(1)}(ka \cos \theta)} \int_{-\infty}^{\infty} e^{-ikz' \sin^2 \theta} Z(z')\, dz' \int_0^{2\pi} \Phi(\phi) \cos m(\phi - \phi')\, d\phi', \tag{24}$$

where we have separated $E_z(a, \phi', z')$ into its longitudinal and transverse variations $Z(z')$ and $\Phi(\phi')$ respectively, i.e.,

$$E_z(a, \phi', z') = Z(z')\Phi(\phi'). \tag{25}$$

Once $Z(z')$ and $\Phi(\phi')$ are specified for a given transverse slot, the far field $E_z(\rho, \phi, z)$ is determined by (24).

Radiation from Thin Transverse Slot. As an application of (24) let us consider a thin transverse slot defined by $|z'| \leq w$, $|\phi'| \leq \phi_0$ with the following distribution across its surface:

$$\begin{aligned} Z(z') &= E_0 \text{ for } |z'| \leq w \text{ where } E_0 \text{ is a constant} \\ &= 0 \text{ for } |z'| > w \\ \Phi(\phi') &= \cos(\pi/2\phi_0)\phi' \text{ for } |\phi'| \leq \phi_0 \text{ where } \phi_0 \text{ is half angle of slot} \\ &= 0 \text{ for } |\phi'| > \phi_0 . \end{aligned}$$

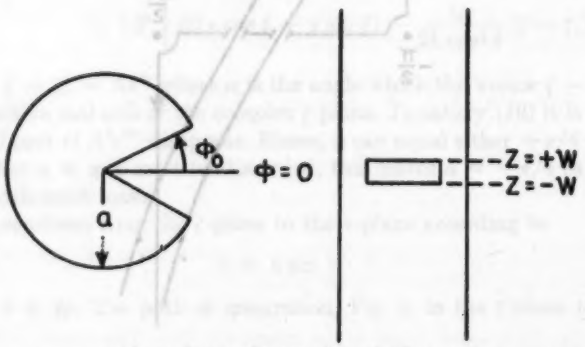

FIG. 5. Top and side views of single thin slot on an infinite cylinder.

The frequency of the source of electromagnetic energy is so chosen that the slot is a half-wavelength wide, i.e. $2\phi_0 a = \lambda/2$ or $\phi_0 = \pi/2ka$. It is clear that

$$\int_0^{2\pi} \Phi(\phi') \cos m(\phi - \phi')\, d\phi' = \int_{-\phi_0}^{\phi_0} \cos(\pi\phi'/2\phi_0) \cos m(\phi - \phi')\, d\phi' = \frac{2ka \cos(m\pi/2ka) \cos m\phi}{(ka)^2 - m^2} \tag{26}$$

and

$$\int_{-\infty}^{\infty} e^{-ikz'\sin\theta} Z(z')\,dz' = E_0 \int_{-w}^{w} e^{-ikz'\sin\theta}\,dz' = 2E_0 \frac{\sin(kw\sin\theta)}{k\sin\theta} \doteq 2wE_0, \tag{27}$$

where the approximation in the final step is valid since $kw \ll 1$. $2wE_0$ is the electromotive force across the slot and is denoted by V. Substituting (26) and (27) into (24) we finally obtain

$$E_z(\rho, \phi, z) = \frac{Vka}{\pi^2} \frac{e^{ikr}}{r} \sum_{m=0}^{\infty} \frac{\epsilon_m e^{-i\frac{1}{2}(m+1)\pi}}{H_m^{(1)}(ka\cos\theta)} \frac{\cos(m\pi/2ka)\cos m\phi}{(ka)^2 - m^2}. \tag{28}$$

If we want the θ-component of the far field, it can easily be shown that $E_\phi = E_z/\cos\theta$, or

$$E_\theta = \frac{Vka}{\pi^2} \frac{e^{ikr}}{r\cos\theta} \sum_{m=0}^{\infty} \frac{\epsilon_m e^{-i\frac{1}{2}(m+1)\pi}}{H_m^{(1)}(ka\cos\theta)} \cdot \frac{\cos(m\pi/2ka)\cos m\phi}{(ka)^2 - m^2}. \tag{29}$$

Results. The far-zone field radiated by a thin transverse slot possessing a cosinusoidal distribution of E_z across its mouth has been calculated from (28) and (29). Using the more general formula (24), it is not difficult to compute the radiation pattern for various combinations of slots. The azimuthal distribution

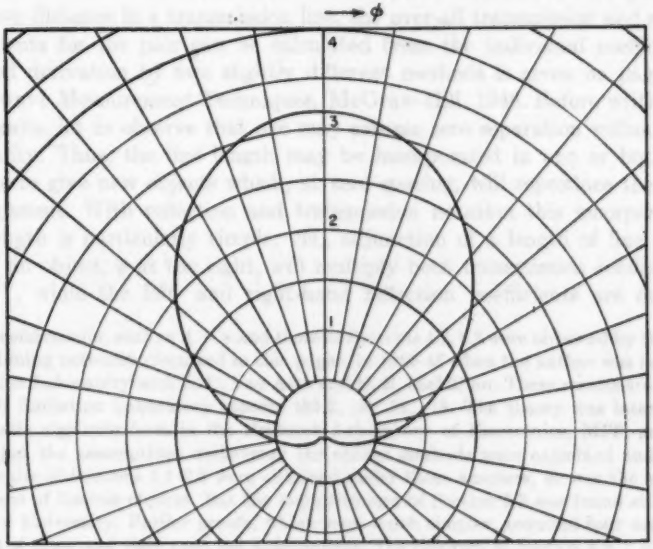

FIG. 6. Radiation pattern in plane $\theta = 0$

for $\theta = 0$ and $ka = 5$ has been computed and is presented graphically[1] in Fig. 6. Recently in a similar but independent paper Pistolkors[3] has derived (29). Noting that he has used $e^{i\omega t}$ instead of $e^{-i\omega t}$ for the harmonic time dependence, the conjugate complex of his equation (33) becomes identical with our (29) after correction of two minor typographical errors in his (33), i.e., the first term in square brackets should be multiplied by i and the sign before the summation should be changed to plus.

Acknowledgment. The author wishes to express his thanks to Dr. H. Levine, of Harvard University, for his generous help.

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(Received December 15, 1948)

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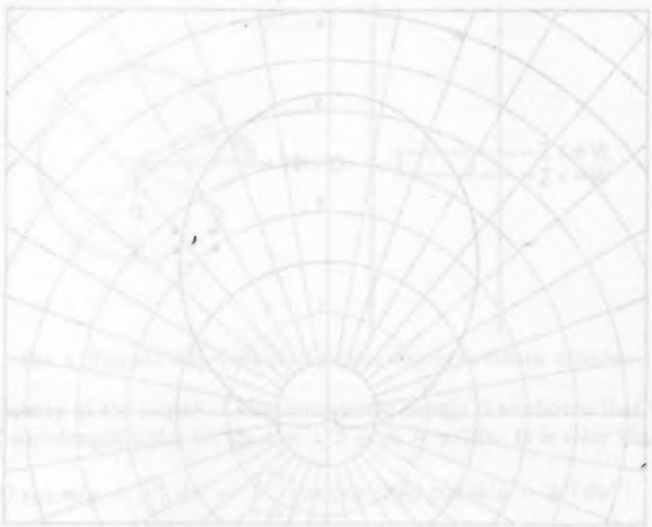
[1] The polar diagram gives the absolute value of the factor multiplying $(Ve^{ikr})/(\pi^2 r)$ in (29) as a function of ϕ.



REMARKS ON THE BASIS OF NETWORK THEORY*

By R. M. Redheffer

1. INTRODUCTION. 1.1 Linearity. We suppose the objects with which we are concerned to be in a transmission line permitting only one mode of propagation at the given frequency, which remains fixed throughout the discussion. The object is said to be *linear from the left* if the complex amplitudes of the transmitted and reflected waves are both proportional to that of the incident wave, whenever this wave is incident from the left. Linearity from the right is similarly defined, and an object is said to be *linear* if it is linear both from the left and from the right. We assume that linearity for a given direction persists even if a wave is traversing the object in the opposite direction.

The above notion of linearity is slightly more general than that usually considered. For example, linearity of this type need not lead to the reciprocity theorem; in other words, the transmission coefficients in the two directions need not be the same. Hence *an object linear in this sense cannot be replaced by an ordinary network* of resistors, inductances, and condensers without additional assumptions. Moreover, if one object of a series is linear from the left only, and the others are linear in both directions, then the whole series, considered as an object, will generally be linear neither from the left nor from the right.

1.2 Transmission and reflection. If two objects, each linear, are separated by a known distance in a transmission line, the over-all transmission and reflection coefficients for the pair can be calculated from the individual coefficients. A detailed derivation by two slightly different methods is given on page 564 of Microwave Measurement Techniques, McGraw-Hill, 1946. Before writing down the results, let us observe that one may assume zero separation without loss of generality. Thus, the line length may be incorporated in one or both of the objects to give new objects which, at zero spacing, will reproduce the original arrangement. With reflection and transmission notation this incorporation of line length is particularly simple, viz., adjunction of a length of line x at the left of an object, y at the right, will multiply both transmission coefficients by $e^{2\pi ik(x+y)}$, while the left- and right-hand reflection coefficients are multiplied

* Special results, such as $A = \pi$ and those of Sections 4.2, 4.3 were obtained by the method of combining networks discussed in this paper, in 1942–45 when the author was in the MIT Radiation Laboratory with L. C. Van Atta and E. B. McMillan. These results are scattered through Radiation Laboratory reports 483-2, -9, -12, -18. The theory was later put into systematic algebraic form in the Research Laboratory of Electronics, MIT, and it was there that the assumptions underlying the earlier methods were examined and isolated. The results of Sections 1.1–2.3 were obtained under these auspices, as was the systematic treatment of lossless objects. But the main theorem of Section 3.3 was found still later at Harvard University. Earlier proofs, which were much simpler, required four assumptions instead of three and they were not independent. The theorem of Section 3.5, $f = F(A, B)$, was also obtained at Harvard as were all results on characterization of passive objects.

respectively by $e^{4\pi ikz}$ and $e^{4\pi iky}$. Here k is the propagation constant of the line (cf. Stratton, Electromagnetic Theory.) Summarizing these remarks, we conclude that *a general theory of adjacent objects in lossless line will include the theory of spaced objects in lossy line.*

We write t, τ for the left- and right-hand transmission coefficients, and r, ρ for the corresponding reflection coefficients. If immediately at the right of an object with these coefficients is placed a second with coefficients t_1, τ_1, r_1, ρ_1 respectively, then the left-hand coefficients of the combination will be $tt_1/(1 - r_1\rho)$ and $(r + dr_1)/(1 - r_1\rho)$, where $d = t\tau - r\rho$, with similar results for the right-hand coefficients.

In the course of deriving these relations one finds incidentally that the transmission and reflection coefficients for the new composite object certainly exist, that is, the relevant ratios are constant. The new object, then, is linear in our sense. By regarding the two objects as themselves forming an object, one obtains the corresponding result for three; by treating the three as a single object the result is extended to four, and so on. In summary, *if each object of a series is linear then the whole series, considered as an object, is also linear*, and the coefficients can be computed by successive use of the above relations.

1.3 Abstract approach. Let us define the symbol $(t\tau r\rho)$ to mean an object with transmission coefficients t, τ and reflection coefficients r, ρ, Arabic and Greek letters being used as above for the left- and right-hand coefficients respectively. We suppose further that the product $(t\tau r\rho)(t_1\tau_1 r_1\rho_1)$ represents the composite object consisting of the original one immediately followed at the right by a second object having coefficients t_1, τ_1, r_1, and ρ_1. With this understanding we see from Sec. 1.2 that the symbols in question must multiply by the rule

$$(t\tau r\rho)(t_1\tau_1 r_1\rho_1) = \left(\frac{tt_1}{1 - r_1\rho}\,\frac{\tau\tau_1}{1 - r_1\rho}\,\frac{r + dr_1}{1 - r_1\rho}\,\frac{\rho_1 + d_1\rho}{1 - r_1\rho}\right) \tag{1}$$

where

$$d = t\tau - r\rho; \qquad d_1 = t_1\tau_1 - r_1\rho_1 \tag{2}$$

and where the right-hand member is a symbol of the same type. If $r_1\rho = 1$ the product is defined as $\lim_{a\to 1} (t\tau r\rho)\,(aa00)\,(t_1\tau_1 r_1\rho_1)$ when this limit exists.

Though perhaps in somewhat bizarre notation, Eq. (1) represents essentially standard results and, as noted in Sec. 1.2, it can be derived by well-known procedures which we omit here. The only new feature is our failure to take $t = \tau$, and, as far as the derivation is concerned, this increase in generality is trivial.

What we propose to do now is to imagine the genesis (as it were) of Eq. (1) to be blotted out, and confine our attention exclusively to the rule of operation there expressed. Let us not think of the symbol $(t\tau r\rho)$ in terms of its origin as an electrical transducer, but rather as a quadruple of complex numbers. These quadruples, whatever they are, will multiply in accordance with (1). Reflecting along these lines we are led to the following definition: *A linear network is a*

quadruple of complex numbers which combines with other quadruples in the manner prescribed by (1). Here we have an abstract algebraic characterization of the notion of network, suitable for the ends in view.

2. ALGEBRAIC PROPERTIES. 2.1 The general group. The assertion that a set of elements forms a group may be regarded as a convenient shorthand for a number of subsidiary assertions. The latter include not only those, such as existence of an inverse, which may have been used in the proof that one has a group, but they include also all the results of group theory. The notation of the previous discussion allows us to use this shorthand procedure, subsuming a multitude of results under a single one. We shall content ourselves here merely with showing that certain sets of objects form groups, without pursuing the implications.

It is clear physically that one will get the same reflection and transmission coefficients for three objects by calculating those for the second and third together, then adding the first, as by calculating those for the first and second together, then adding the third. Mathematically this statement is equivalent to saying that *the multiplication symbolized by Eq.* (1) *is associative*, and direct computation shows that such is indeed the case.

Adding an $n\lambda$ length of line to a lossless network does not change the transmission or reflection coefficients, as is well known. The transmission of such a piece of line is unity and its reflection is zero; presumably, therefore, *the object* (1100) *functions as an identity*: $(1100)(t\tau\tau\rho) = (t\tau\tau\rho)(1100) = (t\tau\tau\rho)$. This relation too is easily verified from (1). After a little calculation, moreover, one finds that the identity is unique in the following sense: If an object $(t\tau\tau\rho)$ functions as either left or right identity for but a single object with nonvanishing transmission, then $(t\tau\tau\rho) = (1100)$.

Proceeding next to the question of inverse we may verify by actual trial or otherwise that, in either order,

$$(\tau/d \; t/d \; -\tau/d \; -\rho/d)\,(t\tau\tau\rho) = (1100) \tag{3}$$

and hence that *any object* $(t\tau\tau\rho)$ *has an inverse the left-hand factor of* (3) provided the determinant $d = t\tau - \tau\rho$ is not zero.

The set of all objects with $d \neq 0$ does not form a group, since for $\tau_1\rho = 1$ the product (1) is not always defined. Also the condition $d \neq 0, d_1 \neq 0$ does not insure the corresponding condition for the product. The above remarks show, however, that *any closed sets of objects with* $d \neq 0$, *and only these sets, form groups under the multiplication* (1). Here closure means that the product of two objects in the set is again in the set.

2.2 Subgroups. Objects satisfying the reciprocity theorem, i.e., objects for which $t = \tau$, are termed *bilateral*. If each of two objects is bilateral, so that $t = \tau$, $t_1 = \tau_1$, then it is clear from (1) that the composite object consisting of the two together will likewise have this property. Proceeding thus we prove by induction that *if each object of a series is bilateral so is the series as a whole*. Observe next from

Sec. 2.1 that the identity is bilateral, as is the inverse of any bilateral object. Combining these remarks we conclude: *If a set of objects forms a group under multiplication* (1), *then the bilateral objects of the set form a subgroup.* Similarly, *in any group the objects with real coefficients form a subgroup, and so do the objects in shunt,* as is shown below.

Hitherto we have considered subsets of groups, without actually exhibiting the group itself. An example of a group is furnished by the results of Sec. 3.3, which show that *the set of lossless objects forms a group under multiplication* (1).

Another example occurs in the problem of finding objects which commute with each other, that is, for which the pair has the same properties independently of order in the line. Equating corresponding members on the right of (1) as it stands and as it becomes when the left-hand factors are interchanged, one finds that a necessary and sufficient condition for the two objects in (1) to commute is $r/\rho = r_1/\rho_1$ and $(1 - d)/r = (1 - d_1)/r_1$. These conditions persist in the products and inverses. Hence if a and b are any constants, *a closed set of objects for which* $r/\rho = a$ *and* $(1 - d)/r = b$ *forms an abelian group under multiplication* (1). Moreover, *if an object commutes with any element of this group other than the identity, then it must itself have* $r/\rho = a$, $(1 - d)/r = b$. Here we have a complete characterization of those objects which give the same results independently of their order. As special cases are symmetric objects with $d = 1$, and any objects with $r = 0$ or $\rho = 0$.

These remarks on groups actually include many results. From the basis theorems of group theory, for example, one might obtain interesting results on the possibility of constructing arbitrary networks, of prescribed type, by using those n a fixed fundamental set.

2.3 Extensions of multiplication. Analyzing the notion of multiplication as symbolized by (1), we find it to depend on the existence of network parameters whose values for two objects together involve only their values for the single objects separately. It has not yet been possible to define such a multiplication with two variables, but for three we have the system

$$(\theta r \rho)(\theta_1 r_1 \rho_1) = \left(\frac{\theta\theta_1}{1 - r_1\rho} \frac{r + dr_1}{1 - r_1\rho} \frac{\rho_1 + d_1\rho}{1 - r_1\rho} \right) \tag{4}$$

where $\theta = \sqrt{1 - r\rho}$, $d = \theta^2 - r\rho$; and the system

$$[r\rho d]\,[r_1\,\rho_1\,d_1] = \left[\frac{r + dr_1}{1 - r_1\rho} \frac{\rho_1 + d_1\rho}{1 - r_1\rho} \frac{dd_1 - \rho_1 r}{1 - r_1\rho} \right]. \tag{5}$$

The possibility of multiplication (4) is linked with a procedure for extending methods originally confined to bilateral networks. The procedure in question is most briefly described by the identity

$$\Pi(l_k\, r_k\, r_k\, \rho_k) = \left(\Pi \sqrt{\frac{l_k}{r_k}} \Pi \sqrt{\frac{r_k}{l_k}}\, 0\, 0 \right) \Pi(\sqrt{l_k r_k}\, \sqrt{l_k r_k}\, r_k\, \rho_k) \tag{6}$$

which can be proved by induction. The left-hand product represents a series of general objects, while the continued product on the right contains only bilateral

objects. Hence *a formula for transmission and reflection of a series of bilateral objects can be used, without essential change, in the general case.* This result is important in that some of the multiplications here enumerated, though in many respects more convenient than (1), do not distinguish the bilateral and nonbilateral cases. For practical computation, Eq. (6) shows that this is no disadvantage. It is true, however, that the form (1) is more fundamental for study of the foundations.

None of these operations is isomorphic to matrix multiplication, but such an isomorphism can be reached by appropriate changes of variable. Specifically, if we define $T = 1/t$, $R = r/t$, $\mathrm{P} = -\rho/t$, $D = d/t$, characterizing the object by $TRPD$ rather than by $t r \rho$, then *the rule of multiplication for these new quantities is equivalent to matrix multiplication*:

$$\begin{pmatrix} D & R \\ \mathrm{P} & T \end{pmatrix} \begin{pmatrix} D_1 & R_1 \\ \mathrm{P}_1 & T_1 \end{pmatrix} = \begin{pmatrix} DD_1 + R\rho_1 & DR_1 + RT_1 \\ \mathrm{P}D_1 + \mathrm{P}_1 T & \mathrm{P}R_1 + TT_1 \end{pmatrix}. \tag{7}$$

Thus, if the entries of the left-hand matrices in (7) give D, R, P, T for two objects separately, then the entries of the matrix on the right give these same quantities in order for the new object consisting of the first two placed adjacently in the line. As is the case for (4) and (5), here too the proof is immediate from (1). This change of variable and the corresponding matrices were suggested to the author by a report of P. Marcus[1]. One may use τ instead of t as the divisor in defining the new variables.

The matrices are defined if $t \neq 0$, and the determinant for a given matrix is equal to τ/t. Hence there is an inverse if $\tau \neq 0$. From (1) the condition $t\tau \neq 0$ in each factor leads to the same condition in the product; and combining these observations we find that *the set of objects with* $t\tau \neq 0$ *forms a group under multiplication* (7). This group, which includes all nontrivial networks, is more extensive than any that could be defined under multiplication (1). The previous results are valid here too; in particular the sets of bilateral and lossless objects are subgroups.

If we divide by $\sqrt{t\tau}$ instead of t or τ, in defining the new variables T, R, P, D, then we obtain other matrices with the required multiplication properties. Here the determinant is always equal to unity, while in (7) it is unity when and only when the object is bilateral. Hence with proper choice of the parameters specifying the electrical properties, *the group of all objects having* $t\tau \neq 0$ *is homomorphic to the group of all bilateral objects having* $t\tau \neq 0$. In a sense this gives the underlying reason for existence of an identity like (6).

Following a procedure due to P. Marcus, we express the general second-order matrix as a linear combination of the so-called Pauli matrices. If these latter matrices be multiplied by i, they combine like the unit elements in quaternion algebra. Applying this method to (7) we find finally that *a network specified by* $a + bi + cj + dk$, *where* $a = (1 + d)/2t$, $b = -(r + \rho)/2t$, $c = (r - \rho)/2it$, $d = (d - 1)/2it$, *will combine with other networks so specified according to the rule*

[1] The Interaction of Discontinuities on a Transmission Line, Radiation Laboratory Report 930, Feb. 6, 1946.

for quaternion multiplication. In other words the coefficients of $1, i, j, k$ in the quaternion product give $(1 + d)/2t, \cdots$ for the new network consisting of the first object immediately followed by the second. The length of the quaternion is unity when and only when $t = \tau$.

In summary, we have found that networks can be combined with systems of multiplication other than (1). Such systems can be used, as we used (1), to define the notion of network. Confining ourselves to objects with nonvanishing transmission, we thus conclude: *The set of linear networks with* $t\tau \neq 0$ *is abstractly equivalent to the set of* 2×2 *matrices with complex entries and nonvanishing determinant, or to the set of nonzero quaternions with complex coefficients.* The corresponding definition for bilateral objects makes the determinant, or the absloute value of the quaternion, equal to unity.

2.4 Matrices. One should observe that the matrix in the above actually stands for the network, in the same sense as does the symbol $(t\tau\rho)$ of (1); it is not used to represent a linear transformation. Because of this meaning, it might be termed the intrinsic matrix of the object. These matrices are useful in that a single product, for a series of objects, gives all the coefficients for both directions.

Of course matrix methods are frequently employed in circuit theory, particularly for problems involving a series of elements. In the better known methods, however, the matrix is often regarded, not as an abstract characterization of the network, but as a linear transformation. The current and voltage at one side of the network, for example, are related linearly to those at the other, and this linear relation may be written in matrix form. The matrix now represents a transformation of the amplitudes in question. As an example, the derivation of impedance relations is often carried out in this way.

Considering this method of introducing matrices in the present transmission-reflection approach, we find that the quantities selected for transformation should be, not current and voltage, but rather the amplitudes of the electric vector. Following P. Marcus (loc. cit.) we resolve the field at the left of the object into two plane waves traveling in opposite directions and having amplitudes A, B. If the object is linear in the sense used here the corresponding amplitudes A_1, B_1, at the right will be linear combinations of A, B; and the methods used to derive (1) show that the linear relation may be written in matrix form as follows:

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} D & R \\ \mathrm{P} & T \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{8}$$

In view of its derivation and significance, the matrix in (8) might be termed the transformation matrix associated with the object.

There is still another means by which matrices can be introduced into circuit theory. If at the left of an object with reflection coefficient x_k is placed a general object $(t_k\tau_k r_k \rho_k)$, then the over-all reflection x_{k+1} is

$$x_{k+1} = \frac{r_k + d_k x_k}{1 - \rho_k x_k} \tag{9}$$

as we see from (1). Thus, the reflection of a series can be expressed as the solution of a difference equation. Following a suggestion of N. Wiener, we replace the unknowns x_k by u_k/v_k and equate numerator and denominator to obtain

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} d_k & r_k \\ -\rho_k & 1 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix}. \tag{10}$$

In this way the solution of the difference equation is expressed by a matrix product. We define $1/t_k$ times the matrix in (10) as the difference matrix of the object, in view of its connection with the difference equation. (The factor $1/t_k$, needed later, is irrelevant in this connection, since we are interested only in the ratio u_k/v_k).

These three devices represent major approaches by which matrices are introduced into circuit analysis. It must be emphasized that the conceptual nature of the matrix, i.e., its interpretation, is quite different in each case. But inspection of the results shows, remarkably enough, that the matrices themselves are the same; in other words, for a linear four-terminal network characterized by appropriate functions of the transmission and reflection, *the intrinsic, transformation and difference matrices are all equivalent.*

3. CHARACTERIZATION PROBLEMS. 3.1 Shunt object. In ordinary circuit theory an object is in shunt when it is connected directly across the transmission line. Of course no such geometric picture can be used here, and the question arises, how shall we define the word *shunt* in the present work?

The cardinal feature of a shunt circuit element is that the voltages just at the left and just at the right are the same. That these two points coincide is incidental. We therefore say here that an object backed by a reflection r_1 is in *temporary shunt* if the E fields at each side of the object are equal. Following through the derivations cited in Sec. 1.2 we find that this equality of fields is equivalent to the condition $r_1(d - t - \rho) + (1 + r - t) = 0$ for the system $(t\tau r\rho)$ $(r_1{}^{***})$. Here the *'s mean that the corresponding coefficients are irrelevant.

From this condition we see that an object can be in temporary shunt for one load r_1 and for no others, or for a wave traveling from left to right but not from right to left. Also every object with $d \neq t + \rho$ is in temporary shunt for some terminating reflection r_1. It is these properties which suggested the phrase "temporary shunt".

Considering the above equation in detail, we find that *the following statements are equivalent*:

i. An object $(t\tau r\rho)$ is in temporary shunt for two different values of the terminating reflection r_1.

ii. Backed by a matched load each time, it is in temporary shunt for waves traveling in either direction.

iii. $t = 1 + r$ and $\tau = 1 + \rho$.

iv. It is in temporary shunt for all values of r_1 and for waves in either direction.

In particular, any statement i, ii, or iii is sufficient to establish iv. We can now

give the required definition, viz., *an object is in shunt when it satisfies one of the above conditions i – iv.*

The well-known statement iii completely characterizes the class of shunt objects in terms of their coefficients. It tells us too that $-\pi/2 \leq \underline{/t} \leq \pi/2$, and that if $\underline{/t} = 0$ then $\underline{/r} = \pi$, when the object is in shunt and both $|t|$ and $|r|$ are ≤ 1. Also any bilateral shunt object has $r = \rho$. Finally, this condition shows that the shunt objects form a subgroup, as asserted previously.

3.2 Definition of lossless objects. In ordinary network theory a network is said to be lossless if all the incident power is either reflected or transmitted, so that none is absorbed. Since the power in a wave of amplitude A is proportional to $|A|^2$, and since our coefficients stand for amplitude ratios, the above condition leads to the well-known relations

$$|t|^2 + |r|^2 = 1, \qquad |\tau|^2 + |\rho|^2 = 1 \tag{11}$$

for waves traveling from left to right and from right to left, respectively.

Ordinarily, when two objects are lossless, so is the composite object formed by placing them adjacently in the line. If this condition is to be preserved here (as it clearly ought) then we cannot define as lossless all objects satisfying (11); for (11) is not preserved by multiplication (1). In addition to (11), therefore, we require that the product of two lossless objects should again be lossless. For understanding of the sequel it is important to observe that this requirement is not the same as saying merely that the product satisfies (11); it says more. The latter condition is too weak to lead to interesting consequences.

If an object is lossless so is the object formed by turning it end for end in the line. This condition, it turns out, is not a consequence of the other two, even though these latter are symmetric conditions. It is added, therefore, as a separate requirement.

One might think that conditions of the above type could be multiplied without end, leading to greater and greater specialization. Actually, however, we shall see that, in a sense which may be precisely specified, we have gone far enough. Hence we now frame our definition, viz., *the set of lossless objects is a complete set satisfying the following conditions*:

i. $|t|^2 + |r|^2 = |\tau|^2 + |\rho|^2 = 1$

ii. If two objects are in the set so is their product

iii. If $(t r r \rho)$ is in the set so is $(\tau t \rho r)$

Completeness of the set means that no new object can be added to it without upsetting one of the properties i–iii. Conditions i and ii are clear; condition iii tells us that an object turned end for end remains lossless. We remark that the second condition in i may be deduced from the first condition and iii.

In constructing this definition we noted at each step that the new condition could not be deduced from the previous ones. Actually more is true, namely, no one condition can be deduced from the other three. In other words, *the requirements i–iii are independent.* For proof, observe that the set of passive networks (as usually defined) satisfies all but i; the set of objects $(ttrr)$ with $|t|^2 + |r|^2 = 1$ satisfies all but ii; and the set $(a00b)$ and $(cc00)$ with $|a| = |b| = |c| = 1$ satisfies all but iii. To finish the proof one simply completes the sets in question,

adding all objects which do not spoil the required properties. This procedure is of no moment for the first example, and for the others we use Sec. 3.3 to show that the completed set will still fail to satisfy the omitted requirement.

3.3 Identification of lossless objects. We have defined lossless objects in terms which might be called "set properties", that is, which involve not only the object itself, but also its relation to other objects in a certain set. Given a specified object, then, there is no procedure for deciding from the definition whether or not it is lossless. The question can be settled only by combination with other objects of the set, and as yet these other objects themselves are specified with the same uncertainty.

One is led to inquire too whether there is but one set of lossless objects. It might happen that two or more sets, each satisfying the conditions i–iii, are such that we could upset these properties in a given set by adding an object from one of the others.

Previously we considered objects which commute under multiplication (1), that is, which satisfy a certain set property. We then deduced that such objects are identical with those for which $r/\rho = a$ and $(1 - d)/r = b$, that is, with those satisfying certain individual properties. The objects in question were thus completely identified. It is our purpose here to do the same thing for the set of lossless objects, and thus to answer the questions posed above. The fundamental result is that *there is only one set of lossless objects, and this set is identical with the set for which simultaneously*

iv. $|t|^2 + |r|^2 = |\tau|^2 + |\rho|^2 = 1$

v. $d = -e^{i(\underline{/\tau} + \underline{/\rho})}$

vi. $|t| = |\tau|$.

When $\rho r = 0$, so that $\underline{/r}$ or $\underline{/\rho}$ is not defined, δ means that $|d| = 1$. It is a simple matter, incidentally, to show that the requirements iv–v are independent and that vi follows from them.

For the first step of the proof, one verifies that the set satisfying iv–vi here also satisfies i–iii of the preceding section. Though long, the calculations are routine, and we omit them. It suffices, then, to proceed in the opposite direction, viz., from i–iii there to iv–vi here.

One finds by actual trial that considering products of two factors, e.g. $(t\tau\tau\rho)^2$, is not sufficient, and that three are probably not sufficient either. We therefore turn at once to $(t\tau\tau\rho)^n$, which, by induction on condition ii, must satisfy i. Equations (27) and (28) thus lead, after simplification, to

$$\begin{aligned}
&\cos 2nu + |t\tau| \cosh(2n-2)u - \cos 2nv \\
&-2\sqrt{|t\tau|}\,[\cosh(2n-1)u \cos x \cos v \\
&+ \sinh(2n-1)u \sin x \sin v - \cosh u \cos(2n-1)v \cos x] \\
&-|t\tau| \cos(2n-2)v = \\
&\quad (|t|^{n+1}/|\tau|^{n-1})(\cosh 2u - \cos 2v) + |r|^2 \quad (\cosh 2nu - \cos 2nv),
\end{aligned} \tag{12}$$

[3] Edwin M. McMillan, Violation of the Reciprocity Theorem in Linear Passive Electromechanical Systems, Jl. Acous. Soc. of America, Vol. 18, p. 344, Oct. 1946.

where $2x = \underline{/t} + \underline{/\tau}$, $\alpha = u + iv$ in (28), and $\sinh \alpha \neq 0$. If $\sinh \alpha = 0$ the same procedure with (31) gives

$$|t|^{n+1}/|\tau|^{n-1} + |\tau|^2 n^2 = n^2 \pm 2n(n-1)\sqrt{|t\tau|}\cos x + |t\tau|(n-1)^2. \quad (13)$$

It is supposed here that $t\tau \neq 0$. If $t = 0$, say, we consider $(\tau 0 \rho r)(0 \tau r \rho) = (00\rho\rho)$, which belongs to the set by ii and iii, and hence satisfies i. Thus $|\rho| = 1$, and therefore $|\tau| = 0 = |t|$. Condition v is evident here, and the result is proved. When $t\tau \neq 0$, as assumed henceforward, the proof is best presented in several stages.

(a) If $u = 0, v = n\pi$, then $|t| = |\tau|$ and $1 + d = 2\cos xe^{iv}$.

If $|t| > |\tau|$, letting $n \to \infty$ (13) gives a contradiction, since $x^n/n^2 \to \infty$ for $x > 1$. If $|t| < |\tau|$ we write (13) with t and τ interchanged, as may be done by iii. Hence $|t| = |\tau|$. When such is the case, the coefficient of n^2 in (13) vanishes, and hence that of n must also. Thus, $2|t|\cos x \pm 2|t|^2 = 0$, which leads to the condition required, upon substitution in (28), if we note that $u = 0, v = n\pi$ makes $\cosh \alpha = \pm 1$, and that $\sqrt{t\tau} = |t| e^{iv}$.

(b) If $u \neq 0$ or $v \neq n\pi$, then $|t| = |\tau|$.

The coefficient of $|t|^{n+1}/|\tau|^{n-1}$ in (12) is not equal to zero with the assumed values of u and v. Denoting this coefficient by A and that of $|\tau|^2$ by B, we observe that A, B and the left side of (13) are unaltered when t and τ are interchanged. Hence the equation analogous to (12) for $(\tau t \rho r)^n$, which one may write by iii, combines with (12) to give $A|t|^{n+1}/|\tau|^{n-1} + B|\tau|^2 = A|\tau|^{n+1}/|t|^{n-1} + B|\rho|^2$. Now if $|t| \neq |\tau|$, it is clear that one side approaches ∞ while the other stays finite; and therefore $|t| = |\tau|$.

(c) If $u = 0$ and $v \neq n\pi$, then $1 + d = 2\cos xe^{iv}$.

In the assumed case Eq. (12) may be written, with $|t| = |\tau|$, $|t|$ $[\cos(2n-2)v - \cos 2v + \cos 2nv - 1] = 2\cos x\,[\cos(2n-1)v - \cos v]$, which becomes $|t|\cos v = \cos x$, if we use the formula for $\cos A + \cos B$ on both sides, divide by $\sin nv$, use the formula for $\sin A + \sin B$ on the left of the result, and divide by $\sin(n-1)v$. Since $\cosh \alpha = \cosh iv = \cos v$, Eq. (28) gives the desired result as before.

(d) If $\lim_{n\to\infty}(a\cos 2nv + b\sin 2nv + c) = 0$ for $v \neq n\pi$ then $a = b = c = 0$.

If v/π is irrational we can take an infinite sequence of n's for which $\cos 2nv \to 1$, as is well known. On this sequence $\sin 2nv \to 0$, whence we conclude $\lim b = -c$, which in turn gives $b = -c$, both being independent of n. Similarly, $b = +c$, so that $b = c = 0$, whence $a = 0$.

On the other hand, if $v/\pi = p/q$, then a sequence $n = kq$ makes $a = -c$, whence the equation may be written $\sin nv[b\cos nv - a\sin nv] \to 0$. Give to n the values $2kq + 1$ to find $b\cos p\pi/q - a\sin p\pi/q \to 0$, therefore $= 0$; and similarly $b\cos 2p\pi/q - a\sin 2p\pi/q = 0$. Regarding these as linear equations in a and b, we note that the coefficient determinant reduces to $\sin p\pi/q$, which $\neq 0$. Hence $a = b = 0$.

(e) If $u \neq 0$ then $v = n\pi$.

We suppose $u > 0$; the case $u < 0$ is similar. Equation (12) involves terms in $e^{2nu}, e^{nu}, e^{0}, e^{-nu}, e^{-2nu}$, each having a different order of magnitude. Dividing

by e^{2nu} and letting $n \to \infty$, we see that the coefficient of e^{2nu} must tend to zero. This coefficient is, however, independent of n, hence actually equals zero. Similarly, the coefficient of e^{nu} is zero. The dominant term is now that not containing e, and it too must tend to zero. This term has the form considered in (d) above with $c = -(t/2)(\cosh 2u - \cos 2v)$; and hence $\cosh 2u = \cos 2v$ whenever $v \neq n\pi$. But the latter equation implies $u = 0$, contrary to hypothesis, since $\cos 2v \leq 1$.

(f) If $u \not\equiv 0$ then $1 + d = 2 \cos x\, e^{iz}$.

By (e) we may assume $v = n\pi$. When this is the case and $|t| = |\tau|$, Eq. (12) becomes, with $|r|^2 = 1 - |t|^2$ and simplifications, $|t|[\cosh(2n-2)u - \cosh 2u + \cosh 2nu - 1] = \pm 2 \cosh(2n-1)u \cos x \mp 2 \cosh u \cos x$. If $u > 0$, dividing by e^{2nu} and letting $n \to \infty$ gives $|t|(e^{-2u} + e^{-u}) = \pm 2e^{-u} \cos x$, or $t \cosh u = \pm 2 \cos x$, which leads to the desired result in view of Eq. (28) and $v = n\pi$.

(g) If $1 + d = 2e^{iz} \cos x$ then $|d| = 1$, $\underline{/d} = (2n+1)\pi + \underline{/\tau} + \underline{/\rho}$.

For $1 + |d| \cos \underline{/d} = 2 \cos^2 x$, $|d| \sin \underline{/d} = 2 \sin x \cos x$, whence, solving the second for $|d|$ and substituting in the first, $\cot \underline{/d} = \cot 2x$, so that $\underline{/d} = 2x + n\pi$, whence $|d| = \pm 1$ by either equation. Since $|d| > 0$, we see that n is even and $|d| = +1$. Finally, $|d|^2 = |t\tau - r\rho|^2 = |t|^4 - 2|rt|^2 \cos A + |r|^4$ with $A = \underline{/t} + \underline{/\tau} - \underline{/r} - \underline{/\rho}$; and for this to be 1 with $|t|^2 + |r|^2 = 1$ we must have $A = (2n+1)\pi$, since then the expression is maximum, and $= 1$. Therefore $\underline{/d} = 2x = \underline{/t} + \underline{/\tau} = \underline{/r} + \underline{/\rho} + (2m+1)\pi$ and the present result follows. This completes the proof of the main theorem also.

3.4 Other approaches. In ordinary network theory if one side of a lossless object is matched to the line by a lossless tuner, then the other side will be matched automatically, and adding a suitable length of line makes the phase shift on transmission equal $2n\pi$, so that the transmission coefficients are both unity. The combination of line plus tuner plus object, then, is (1100), whence we see that a lossless object has a lossless inverse. These remarks suggest the following result, the word *lossless* now being understood in the sense of this article: *The set of lossless objects coincides with the complete set* $(t\tau r\rho)$ *for which*

vii. $|t|^2 + |r|^2 = 1$, $|\tau|^2 + |\rho|^2 = 1$

viii. If two objects are in the set so is their product

ix. If an object is in the set so is its inverse.

Recall by (3) that the inverse of $(t\tau r\rho)$ is $(\tau/d\ t/d\ -r/d\ -\rho/d)$. Since this is in the set it satisfies vii, hence $|\tau|^2 + |r|^2 = |d|^2, |t|^2 + |\rho|^2 = |d|^2$. Combination with vii gives $|t| = |\tau|$. Since the only use of iii in the previous proof was to show $|t| = |\tau|$, we may use that proof to conclude that here too $d = -e^{i(\underline{/r} + \underline{/\rho})}$. Thus vii – ix imply iv – vi, which in turn imply i – iii.

It may be mentioned that using $|d| = 1$ on $(t\tau r\rho)^2$ gives $|d^2 - r\rho| = |1 - r\rho|$, whence either $r = 0$ or $\underline{/d} = n\pi$ or $\underline{/d} = \underline{/r} + \underline{/\rho} + n\pi$. A proof independent of the preceding could be constructed along these lines.

To show independence observe that the last example in Sec. 3.2 satisfies all

but ix; the set with $|d| = 1$ satisfies all but viii; and the set of shunt objects satisfies all but vii.

In view of the associative character of our multiplication, the result just obtained could be worded as follows: *If a set of objects satisfying i forms a group, then it must be a subgroup of the group of all lossless objects.*

Next, the objects (00 −1 −1), (0011), and (aa00) with $|a| = 1$ satisfy iv–vi. Hence *a short circuit, an open circuit, and an arbitrary length of lossless line are all lossless.* This result, altogether trivial in ordinary theory, perhaps illustrates the nonstandard character of the present treatment. It means that if any set of objects satisfies i–iii, then the enlarged set formed by adding the above will also satisfy i–iii. When the result is thus stated its self-evidence fades a little, and as a matter of fact it can only be proved by an argument analogous to that of Sec. 3.3.

In ordinary theory a lossless object followed at any distance by a short circuit reflects all the incident power, so that $|r| = 1$ for the combination. This property also implies that the object be lossless, and it persists when the object is turned end for end. Knowing that a short circuit and a lossless line are lossless even in the present sense, we are led to the following theorem: *The set of lossless objects is identical with the complete set* ($t\tau r\rho$) *for which*

x. $|t|^2 + |r|^2 = 1, |\tau|^2 + |\rho|^2 = 1$

xi. Whenever $|a| = 1$, ($t\tau r\rho$) (aa00) (0011) has left-hand reflection unity in absolute value

xii. The same is true for ($\tau l\rho r$)

These requirements are independent.

A similar theorem might be given with (0011) replaced by ($t\tau r\rho$) or its transpose. For the proof write $a = e^{ix}$ and use xi to get an expression which leads to

$$|r|^2 + |d|^2 = 1 + |\rho|^2 \tag{14}$$

$$|dr| \cos(\underline{/d} - \underline{/r}) + |\rho| \cos \underline{/\rho} = 0 \tag{15}$$

$$|dr| \sin \underline{/d} - \underline{/r}) + |\rho| \sin \underline{/\rho} = 0 \tag{16}$$

when we expand the cosines and equate to zero the coefficients of 1, cos $2x$, and sin $2x$. The equation analogous to (14) with r and ρ interchanged combines with (14) to give $|r| = |\rho|$ and $|d| = 1$. Hence if $r\rho = 0$ we have v, vi, and the theorem. If $r\rho \neq 0$ dividing (15) and (16) leads to vi, and thus again to the theorem. Independence follows by considering objects with $|r| = 1$, $|t| = 0$, and $|\tau|^2 + |\rho|^2 = 1$, which satisfy x and xi but not iv–vi. Similarly one shows easily that iv–vi imply x–xii.

A curious feature of this result is that it makes but slight use of the symmetry condition xii; in fact *condition xii may be replaced by*

xiii. If $\rho = 0$ then $|r| = 0$; if $|\rho| = 1$ then $|r| = 1$

and both requirements xiii are needed. For the proof, when $r = \rho = 0$ (14) gives $|d| = 1$ and hence the final result. If $\rho \neq 0$ then (15) and (16) together show $dr \neq 0$, whence by division $\underline{/d} = \underline{/\rho} + \underline{/r} + n\pi$. Using this in (15), (16) and dividing by cos $\underline{/\rho}$ or sin $\underline{/\rho}$ (one of which is not zero) we find that n is odd and

that $|dr| = |\rho|$. Equation (14) now leads to $|r| = |\rho|, |d| = 1$ whenever $|r| \neq 1$, and thus the proof is complete.

Finally, *the set of lossless objects is identical with the set for which*

xiv. $(efgh) = (t\tau r\rho)$ $(abcd)$ has $|e|^2 + |g|^2 = 1$ whenever $|a|^2 + |c|^2 = 1$

xv. $(abcd)$ $(t\tau r\rho) = (ijkl)$ has $|j|^2 + |l|^2 = 1$ whenever $|b|^2 + |d|^2 = 1$

These requirements are independent.

As usual one readily shows that lossless objects have these properties and that they are independent. By choosing $(abcd) = (1100)$ we get i, whence xiv alone gives $|c|(|r|^2 + |d|^2 - |\rho|^2 - 1) + 2|\rho| \cos(\underline{/c} + \underline{/\rho}) + 2|rd| \cos(\underline{/c} + \underline{/d} - \underline{/r}) = 0$. Since $|c|$ and $\underline{/c}$ are independent variables the coefficient of $|c|$ vanishes; and this relation with ρ and r interchanged, which we have by xv, gives $|r| = |\rho|, |d| = 1$. The other term of the equation now leads immediately to $\underline{/d} = \underline{/r} + \underline{/\rho} + (2n + 1)\pi$ whenever $r\rho \neq 0$, so that the proof is complete. As was the case for xii, condition xv is equivalent to an apparently less restrictive condition.

3.5 Discussion. The distinctive feature of the preceding is that we have only $|t| = |\tau|$, not $t = \tau$. Except for that the class of lossless objects could be proved identical with the class of lossless networks. We propose now to show that this weak reciprocity cannot be improved; in fact *the class of lossless objects cannot be restricted beyond iv–vi by any methods depending on power alone.* Given a lossless object $(t\tau r\rho)$ suppose A and C are the complex amplitudes of waves traveling from left to right on the left and right of the object respectively, while B and D are the corresponding amplitudes for waves traveling from right to left. Using superposition for the amplitudes we have $B = Ar + D\tau$, $C = At + D\rho$. The total power entering the object from the left is $|A|^2 - |B|^2$, and since the object is lossless this must equal $|C|^2 - |D|^2$, the total power leaving at the right. One is thus led to a relation

$$p^2(1 - |r|^2 - |t|^2) - 2p[|r\tau| \cos(\underline{/\tau} - \underline{/r} - \underline{/q}) + |\rho t| \cos(\underline{/\rho} - \underline{/t} - \underline{/q})] + 1 - |\tau|^2 - |\rho|^2 = 0, \tag{17}$$

upon substituting for B, C, dividing by D, writing $A/B = pe^{iq}$, and collecting the coefficients of p^2, p, 1. If we ask that this be true for all p and q, we have made the maximum possible use of superposition involving only the terminals, not the interior. The coefficients of p^2 and 1 give condition iv; that of p makes $|r\tau| = |\rho t|$ and $\underline{/\tau} - \underline{/r} = (2n + 1)\pi + \underline{/\rho} - \underline{/t}$, when we use the fact that q also is a continuous variable. Combined with condition iv just obtained, the first gives v and the second then gives vi. Moreover, nothing else can be deduced; if iv–vi is satisfied then Eq. (17) becomes an identity in both p and q. The only remaining possibility is to consider several objects placed adjacently in the line, and the previous work shows that this also will lead to nothing new.

By the above reasoning, for which the author is indebted to L. J. Chu, we have a simple proof that lossless objects satisfy the conditions iv–vi. It is this method that one should use in ordinary network theory. The complexity of the preceding

proof was due to the unrestrictive character of the assumptions i–iii which, it is thought, represent the minimum that one can reasonably demand of a linear lossless network. The two results together suggest the following: *If a theory of lossless objects is developed in terms of the terminals only, then the weakest demands one may reasonably make lead to a unique set which satisfies the strongest demands one may reasonably make.*

The preceding shows that the possibility of proving $\underline{/t} = \underline{/\tau}$ represents the difference between a theory that assumes its linearity and superposition properties at the terminals only, and one that assumes these throughout the network. Or again it represents the difference between a theory based solely on power and one that also uses energy. From still a third viewpoint, it gives the distinction between networks linear in our sense and those that can be represented as a finite combination of resistances, capacitances, and inductances. Electromechanical systems with $\underline{/t} \neq \underline{/\tau}$ have been actually constructed. One example[2] consists of a dipole connected to a movable coil in a magnetic field. Motion of the coil changes plate separation of a condenser connected in series with a second dipole and a battery. For small fields this circuit is linear and lossless, but gives $t = -\tau$, or $\underline{/t} - \underline{/\tau} = \pi$. In agreement with the present theory, $|t| = |\tau|$. For passive circuits one may even have $|t| \neq |\tau|$ (loc. cit. and see below).

In the course of the preceding proofs it was shown that the condition $d = -e^{i(\underline{/\tau}+\underline{/\rho})}$ is equivalent to $A = (2n+1)\pi$, where

$$A = \underline{/t} + \underline{/\tau} - \underline{/r} - \underline{/\rho}. \tag{18}$$

Henceforward we shall take this condition as simply $A = \pi$. It was shown also that one has reciprocity for the amplitudes only, and thus $B = \underline{/t} - \underline{/\tau}$ gives a measure of the deviation. The angles A and B are fundamental in the present theory. For any object, lossless or not, A and B are *intrinsic* in the sense that if an arbitrary length of line be added to each side of the object, $(t\tau r\rho)$ being replaced by $(aa00)\,(t\tau r\rho)\,(bb00)$ with $|a| = |b| = 1$, then A and B are unchanged. Of course the absolute values $|t|, \cdots$ are also unchanged; and what we propose to show now is that these are essentially the only quantities with that property. Specifically, *if* $f(t, \tau, r, \rho)$ *is any function invariant under addition of line lengths, then* $f = F(A, B)$, where both sides naturally may involve $|t|, |\tau|, |r|, |\rho|$.

For the proof regard the amplitudes as constant and write $f = f_1(\underline{/t}, \underline{/\tau}, \underline{/r}, \underline{/\rho}) = f_1(\underline{/t} + x + y, \underline{/\tau} + x + y, \underline{/r} + 2x, \underline{/\rho} + 2y)$, the latter equation following for any x and y by the assumed independence of line length. Picking $\underline{/t} - \underline{/\tau}$, $\underline{/r} - \underline{/\rho}$, and $\underline{/\rho} - \underline{/\tau}$ as new variables, we get a new function which the above condition with $x = y$ shows independent of $\underline{/\rho}$. Thus $f = f_2(\underline{/t} - \underline{/\tau}, \underline{/r} - \underline{/\tau}, \underline{/\rho} - \underline{/\tau}) = f_2(\underline{/t} - \underline{/\tau}, \underline{/r} - \underline{/\tau} - z, \underline{/\rho} - \underline{/\tau} + z)$, the latter term again resulting from independence. The first may be written $f_2(A, A - B - \underline{/\rho} + \underline{/\tau}, 0 + \underline{/\rho} - \underline{/\tau})$, which is independent of $\underline{/\rho} - \underline{/\tau}$ as we see by using this value for z. It is therefore a function of A and B alone as was to be proved.

The angle A also has a simple geometric meaning. If an object $(t\tau r\rho)$ is followed at the right by a moving object, i.e., by a reflection R with $|R|$ constant but $/R$ variable, then the over-all reflection of the combination traces out a circle in

the Argand diagram, as is well known, and readily proved[3] from (1). If the position of the center of this circle is denoted by the complex number c, then A *is the positive angle from the line* $0 - r$ *to the line* $r - c$. Hence for lossless objects the origin, c, and r, are colinear with c between 0 and r. It is remarkable that the angle in question is independent of R, the moving reflection.

Another geometrical property depends in a simple way on d, namely, *the origin will be inside of, on, or outside of the circle according as* $|r/R| < |d|$, $= |d|$, *or* $> |d|$. Hence the condition for zero reflection with some $\underline{/R}$ is that $|R| = |r/d|$, which is possible with $|R| \leq 1$ when and only when $|r| \leq |d|$. Here we have the necessary and sufficient condition for matching an object at the left by putting a passive tuner at its right. For lossless objects the conditions become the usual ones $|r| >$, $=$, or $< |R|$, which are identical with those obtained from the so-called vector diagram.

Once the properties of lossless objects are found, the relevant equations simplify considerably, as is well known. For two objects one uses the relation in (1) for transmission, which is always simple, and then the reflection is found from i. Besides this facility of computation, proofs of many general theorems can be easily given. For example *if a lossless object takes a match into a match then it takes any standing-wave ratio into the same standing-wave ratio.* For the first condition gives $r = 0$; hence $|t| = 1$; hence $|\tau| = 1$; hence $\rho = 0$; and the result follows. A proof by impedance methods is more difficult.

3.6 Passive objects. By a passive object is meant one that seems to produce no energy, so that the equality in relations like $|t|^2 + |r|^2 = 1$ is replaced by $\leq$. The most restrictive and hence simplest relation (17) becomes the same with the $=$ replaced by $\geq$, whence letting $p \to 0$ or ∞ gives

$$\alpha^2 = 1 - |t|^2 - |r|^2 \geq 0, \qquad \alpha^2 = 1 - |\tau|^2 - |\rho|^2 \geq 0. \tag{19}$$

When this is the case we have a nonnegative quadratic in p, so that the discriminant is ≤ 0. Because q is a continuous variable, this last is equivalent to saying that the maximum of the discriminant (with respect to q) is also ≤ 0, whence finally

$$|r\tau|^2 + 2|r\tau\rho t| \cos A + |\rho t|^2 \leq \alpha^2 \alpha^2 \tag{20}$$

as is seen most easily by noting that the discriminant in question is the real part of $|r\tau| e^{i(\phi+A)} + |\rho t| e^{i\phi}$ with ϕ variable. The condition (20) may also be written $|r\bar{\tau} + \bar{\rho}t|^2 \leq \alpha^2\alpha^2$.

One can give for passive objects a definition similar to i–iii above for lossless objects, and thus construct a theory parallel to the preceding. The condition corresponding to iv–vi is now (19) and (20). Except for vii–ix, which involves the inverse, the different modes of characterization outlined in Sec. 3.4 also have their analogues here; and there is the added theorem that *every lossless object is passive.* Since the methods in an axiomatic program of this sort so closely resemble

[3] See for example E. C. Titchmarsh, Theory of Functions, Oxford University Press, Ch. VI.

the preceding ones, we omit the details. On the other hand there are questions, unimportant before, which now merit closer consideration.

It must be shown that the properties (19) and (20) are preserved by multiplication (1), whence we conclude as before that passive objects cannot be restricted beyond (19) and (20) by any argument depending on power alone. For proof, suppose that two objects each satisfying (19) and (20) are placed adjacent in the line, and that the powers in the waves moving from left to right at the left of, between, and at the right of the objects are A, C, E respectively; and let B, D, F represent the corresponding power from right to left. Since the first object satisfies (19) and (20), we know that $C - D \leq A - B$; and since the second does also, $E - F \leq C - D$. Hence $E - F \leq A - B$, and therefore the pair satisfies the complete hypothesis needed for proof of (19) and (20). Repetition of that proof leads to the desired result.

One can also show by direct calculation that (19) is preserved in the product, as follows. For $(t\tau r\rho)$ $(T\mathbf{T}R\mathbf{P})$, Eq. (19) is at worst

$$2[\,|\rho|^2(1-|\tau|^2)^2 + 2\,|r\rho t\tau|\,(1-|\tau|^2)\cos A + |rt\tau|^2]^{1/2}$$
$$\leq a^2/|R| + |R|\,(|\rho|^2 + |t|^2 + 2\,|t\tau r\rho|\cos A - |r\rho|^2 - |t\tau|^2) \tag{21}$$

as is seen when we compute the left side of (19) by (1), clear of fractions, put all terms involving cosines on the left, maximize these with respect to $\underline{/\rho} + \underline{/R}$, and divide by $|R|^2$. For the maximizing, note that the expression is the real part of a certain exponential sum.

The term on the right of (21) has the form $P/|R| + Q|R|$, which is least when $|R| = \sqrt{P/Q}$ and which is then equal to $2\sqrt{PQ}$. This substitution thus gives an inequality stronger than (21); and carrying it out leads after simplification to (20), so that the proof is complete. We have incidentally given a new proof of (20) itself, which depends only on (19) and the multiplicative properties of the symbols.

For passive networks there is no exact reciprocity even for amplitudes (McMillan, loc. cit.), but we do have $|t| = |\tau|$ when the objects are lossless, that is, when $a = \alpha = 0$. It is natural therefore to expect that $|t|$ and $|\tau|$ will be in some sense close if a and α are small, and such is the case. The proper quantities to compare, however, are not $|t|$ and $|\tau|$, but rather $|r/t|$ and $|\rho/\tau|$, for which Eq. (20) gives immediately

$$\left(\left|\frac{r}{t}\right| - \left|\frac{\rho}{\tau}\right|\right)^2 \leq \left|\frac{a\alpha}{t\tau}\right|^2 - 4\left|\frac{r\rho}{t\tau}\right|\cos^2\frac{1}{2}A. \tag{22}$$

The preceding discussion shows that this inequality cannot be improved. It is least restrictive when $A = \pi$, and on the other hand $A = \pi$ is consistent with (19) and (20). Hence *for a passive object* $(t\tau r\rho)$ *with absorption coefficients* a^2 *and* α^2 *one has*

$$\left|\left|\frac{r}{t}\right| - \left|\frac{\rho}{\tau}\right|\right| \leq \frac{a\alpha}{|t\tau|}, \tag{23}$$

and equality can always be attained. Here we have the analogue of reciprocity for passive objects. It shows in particular that *if a passive object is lossless in either direction* (so that a or $\alpha = 0$) *then* $|r/t| = |\rho/\tau|$ and $A = \pi$.

If an object is lossless the coefficients must satisfy certain relations in addition to $|t| = |\tau|$. For example $A = \pi$ with A given by (18); $|d| = 1$ in (2); and $\underline{/d} = \pi + \underline{/r} + \underline{/\rho}$. Using a^2, α^2, and $b^2 = 1 - |t\tau| - |r\rho|$ as a measure of loss, we inquire whether there are conditions analogous to these for passive objects.

Adding $|t\tau|^2 + |r\rho|^2 - |\tau\tau|^2 - |\rho t|^2 - 2$ to (20) and changing signs gives the right-hand inequality in $0 \leq a^2 + \alpha^2 - (a^2 + \alpha^2)^2/4 \leq b^2(2 - b^2) \leq 1 - |d|^2 \leq a^2 + \alpha^2$. The inequality involving b results from $d \leq |t\tau| + |r\rho|$, and the left-hand term by using $2xy \leq x^2 + y^2$ on this. These inequalities, none of which can be improved, show how far $|d|$ deviates from unity. In particular

$$|d|^2 = 1 - a^2 - \alpha^2 \tag{24}$$

except for terms in a^4 and α^4.

To estimate the deviation of A from π it suffices to rewrite (22), putting the $\cos^2$ on the left and the other terms on the right. It is seen that *the more A deviates from π the more nearly must one have reciprocity*, and conversely. Since the inequality is least restrictive when $|r/t| = |\rho/\tau|$, and since this condition can actually occur, we conclude that $\pi - A$ *satisfies* $4 \sin^2 \tfrac{1}{2}(\pi - A) \leq |a\alpha|^2/|t\tau\rho|$, *and equality can always be attained.* Thus A is unrestricted when and only when $a^2\alpha^2 \geq |t\tau\rho|$. Except for terms in $(a\alpha)^3$ we have

$$|\pi - A| \leq a\alpha/\sqrt{|t\tau\rho|}. \tag{25}$$

Finally, the relation $|d| \sin(\underline{/d} - \underline{/r} - \underline{/\rho}) = |t\tau| \sin A$ combines with the above to give similar inequalities for the deviation of $\underline{/d}$ from $\pi + \underline{/r} + \underline{/\rho}$. Using (24), (25), and the fact that $\underline{/d} = \underline{/r} + \underline{/\rho} + \pi$ for zero loss, one gets in particular

$$|\underline{/d} - (\pi + \underline{/r} + \underline{/\rho})| \leq a\alpha\sqrt{|t\tau/r\rho|}, \tag{26}$$

except for terms a^4, α^4, and $(a\alpha)^3$. This is an interesting complement to (25).

It should be noted that the three equations (19) and $|d|^2 \geq 1 - a^2 - \alpha^2$ completely characterize passive objects as they stand, and lossless objects when made into equalities. All other conditions can be deduced from these, including the relations for phase, even though the equations themselves involve only the amplitude. There is a certain propriety in being able to characterize the objects thus, as the original physical condition was a condition on power alone.

The foregoing relations permit one to estimate the absorption coefficient of two objects in terms of the individual coefficients. As a simple illustrative case, suppose two identical symmetrical objects $(ttrr)$ are spaced for maximum transmission. This amounts to considering $(ttrr)^2$ with r real and positive. Computing the absorption by (1), noting by (26) that $\cos \underline{/d} = -1 + O(a^4)$, using (24) to get $|d| = 1 - a^2 + O(a^4)$, and taking $(1 - |r|^2)^2 = |t|^4 + O(a^2)$ in the denominator, we find that *the over-all absorption of* $(ttrr)^2$ *with* $r > 0$ *is* $2a^2/|t|^2$, except for terms in a^4.

Such questions can also be treated from first principles by another method. If a passive object has left-hand reflection $\rho = 0$, then not only is $|t|^2 + |r|^2 \leq 1$, but $|t\tau| + |r| \leq 1$. For by inspection the maximum reflection of the object backed by a length of line and a short circuit is $|t\tau| + |r|$. The above inequality is only slightly weaker than that given by (20).

Next, suppose $(t\tau r\rho)$ is followed by a lossless tuner with reflection $|\rho|e^{-i\rho'}$. The object is then matched from the right, so that the over-all reflection ρ is zero, and hence the previous result may be used. The product of the transmissions is found by inspection, and shows that the reflection cannot exceed $(1 - |t\tau| - |\rho|^2)/(1 - |\rho|^2)$. We conclude as a special case: *If a bilateral passive object $(tt r\rho)$ is followed by an object with left-hand reflection ρ, and the two spaced for maximum transmission, then the over-all reflection cannot exceed $\alpha^2/|t|^2$.* Besides its practical utility, this result has theoretical interest in that it relates an amplitude reflection on the one hand to power transmission and absorption on the other. Combined with the result of the previous paragraph, it shows that *if two identical symmetric objects be spaced for maximum transmission, then the over-all reflection in amplitude cannot exceed one-half the over-all absorption in power*, plus terms involving the square of this absorption.

The above method, which is general, consists of matching an object, backing it by a similar object, or otherwise modifying it, and deducing its properties from those of the combination This procedure generally leads either to optimum results, or to results differing from optimum only in higher-order terms. It can be used to give a simple proof of reciprocity as follows. If $(t\tau r\rho)$ is followed by $(\tau t\rho r)$ at such a distance as to maximize the over-all transmission, then this maximum transmission is $|t\tau|/(1 - |\rho|^2)$, by inspection of (1). Hence $|t\tau| \leq 1 - |\rho|^2$. Similarly, $|t\tau| \leq 1 - |r|^2$; and if the object is lossless these become $|t\tau| \leq |\tau|^2, |t\tau| \leq |t|^2$, so that $|t| = |\tau|$. As is often true of the method in question, the proof starts from first principles and involves little calculation.

4. SERIES OF REFLECTIONS. 4.1 Remarks on the general case. The different multiplications considered in Sec. 2.3 lead to different expressions for the transmission and reflection of a series of objects. For example the answer may be expressed as a matrix product, as a quaternion product, or as the solution of a difference equation. Since the procedure of calculation is generally well known in both mathematical and engineering literature, only a cursory discussion will be given here. As an illustration one may make the substitution $x_k = y_k + 1/\rho_k$ in (9), which then becomes a difference equation of form $y_{k+1} = a_k + b_k/y_k$; and the solution can be written as a continued fraction by inspection. The substitution $y_k = -\rho_k x_k$ leads to an equation which has been solved in terms of determinants, but the derivation is more difficult.

A new approach due to P. Marcus gives an explicit form as a sum of products. The procedure is to write (10) as two difference equations, then express the solution as a sum of products of the individual coefficients, and finally prove it by induction.

In an alternative proof (P. Marcus, loc. cit.) the transmission is expressed in a compact form which on expansion gives the sum of waves following all possible

multiple reflection paths once, and no other. A single multiple reflection path for a transmitted wave emerging from the last object is called a ray, and a ray which is reflected just twice is called simple. A ray is elemental if it is simple or a combination of simple rays which can have only one order. Associated with each ray is the ray transmission coefficient, which is the ratio of transmitted to incident amplitude on the ray; that is, the product of reflection and transmission coefficients of all objects from which it is reflected or through which it is transmitted, respectively. Dividing this coefficient by the part common to all rays, i.e. by $t_1 t_2 \cdots t_n$, gives the partial transmission coefficient. The partial transmission coefficient of any ray is the product of the partial coefficients of its component simple rays. In terms of these quantities the over-all transmission coefficient T can be expressed as

$$T = \frac{t_1 t_2 \cdots t_n}{1 - \Sigma(-1)^c \text{ (elemental rays)}}$$

where c is the number of simple rays making up each elemental ray, and the elemental rays are represented algebraically by their partial transmission coefficients. This is the compact form mentioned above. If one makes a certain systematic classification of elemental rays, the equation leads to an expression identical with that first found algebraically.

This method is useful on account of the simplicity of the final result, the explicit formulation, and the suggestive nature of the derivation, which allows many problems to be treated diagrammatically. It is the analogue for n objects of the usual multiple reflection methods for two. In the present connection its importance is that it reduces questions of transmission for series of objects to questions in combinatorial arithmetic, and thus suggests a purely combinatorial approach to the foundations of network theory.

4.2 Repeated elements. If the objects of a series are identical then the over-all transmission and reflection can be expressed in simple closed form, as is well known. One procedure[4] is to obtain simultaneous difference equations in two unknowns from (8), combine these into a single equation in one unknown, write the general solution, and substitute the end conditions. With $(t\tau r\rho)^n = (T_n \mathrm{T}_n R_n \mathrm{P}_n)$, the solution (which assumes $t = \tau$) combines with (6) to give

$$\begin{aligned} T_n &= \left(\frac{t^{n+1}}{\tau^{n-1}}\right)^{1/2} \frac{\sinh \alpha}{\sinh \alpha n - \sqrt{t\tau} \sinh \alpha(n-1)} \\ R_n &= \frac{r \sinh \alpha n}{\sinh \alpha n - \sqrt{t\tau} \sinh \alpha(n-1)} \end{aligned} \tag{27}$$

for the general case $t \neq \tau$, where

$$\cosh \alpha = \frac{1+d}{2\sqrt{t\tau}}. \tag{28}$$

[4] Cf. P. Marcus, loc. cit., where reference is also given to Fiske, A Broad Band T-R Switch, G. E. Research Laboratory Report, Oct. 18, 1943, and to Smythe, Static and Dynamic Electricity, p 219, p 336.

Interchanging (t, τ) and (r, ρ) in (27) gives T_n and P_n. It may be noted that Eq. (28), $A = \pi$, and $|d| = 1$ give $\sqrt{|t\tau|}\cos\alpha = \cos\frac{1}{2}(\underline{/t} + \underline{/\tau})$ for lossless objects.

P. Marcus has obtained another proof of (27) by expressing the matrix (10) in terms of Pauli matrices and the variables in terms of trigonometric functions.

It is clear from (1) that $(t\tau r\rho)^n$ is a rational function of t, τ, r, ρ; and it can be thus expressed by (27) or as follows. In Eq. (9) replace x_k by R_k and drop the subscript k on the other terms to get the difference equation for R_k when all objects are equal. The substitution $R_k = 2ry_k/[1 + (1 - d + v)y_k]$, where $v^2 = (1 - d)^2 - 4r\rho$, leads to a new difference equation in y_k, which can be solved as a sum by inspection. Carrying out the summation gives

$$R_n = r\frac{1 - r\rho + vw + t\tau}{1 - r\rho + vw - t\tau} \tag{29}$$

with w determined from

$$\frac{w + 1}{w - 1} = \left(\frac{1 + d + v}{1 + d - v}\right)^{n-1}. \tag{30}$$

The expression is free of radicals, as v will occur only in the square. An alternative derivation of (29) has been obtained from (10) by means of an expression for the n^{th} power of the matrix due to H. M. James. The procedure is to write it as a sum of matrices that commute and use the binomial theorem.

For certain special cases the equations simplify considerably. If $1 + d = 2\sqrt{t\tau}$, for example, then $\alpha = 2k\pi i$, and taking the limit in (27) gives

$$T_n = \left(\frac{t^{n+1}}{\tau^{n-1}}\right)^{1/2}\frac{1}{n - \sqrt{t\tau}(n-1)}, \qquad R_n = \frac{rn}{n - \sqrt{t\tau}\,(n-1)}. \tag{31}$$

There is a similar expression when $1 + d = -2\sqrt{t\tau}$. If $\rho = 0$ we get

$$(t\tau 0)^n = \left(t^n\tau^n r\frac{1 - d^n}{1 - d}\,0\right)$$

by induction on (1) or by considering multiple reflections; and similarly

$$(t0r\rho)^n = \left(\frac{t^n}{(1 - r\rho)^{n-1}}\,0r\rho\right).$$

Many properties of series are best derived directly without intervention of the general equation. For example it is clear physically that the reflection R_n in (27) has a limit as $n \to \infty$ whenever the object is lossy, as then the addition of more objects eventually has negligible effect. To evaluate this limit directly recall (Sec. 2.2) that $R_n/\mathrm{P}_n = a$ and $(1 - D_n)/R_n = b$ for the whole series whenever these equations hold for each object of the series. Here $D_n = T_n\mathrm{T}_n - R_n\mathrm{P}_n$. This invariance of r/ρ and $(1 - d)/r$ gives a general method by which knowing one of the three variables $T_n\mathrm{T}_n$, R_n, P_n allows us to calculate the other two. For the present problem the fact that $T_n\mathrm{T}_n \to 0$ with loss shows that $\lim D_n = -\lim R_n\mathrm{P}_n$; hence by simultaneous solution $\lim 2R_n = ab \pm \sqrt{a^2b^2 - 4a}$.

As a special case this includes series of identical objects, with $b = (1 - d)/r$, and gives the limit of the reflection as the number of objects increases indefinitely.

The vector diagram, which is often used when the reflections are small, gives $R_n \cong \Sigma r_k$, or, for identical objects, $R_n \cong nr$. By the above method, or by considering multiple reflection, one finds the exact result

$$\lim_{r \to 0} \frac{R_n}{r} = \frac{1 - (tr)^n}{1 - tr}.$$

Hence the fractional error in taking $R_n = nr$ is approximately $(n - 1)(1 - tr)/2$, a relation which shows how closely the reflection of the series is approximated by the sum of the individual reflections.

The vector diagram also suggests that two or more identical objects can be equally spaced for zero reflection, as the corresponding figure is then a regular polygon. If there is no loss this property can be proved without approximation by (27) or as follows. Equation (1) shows that two lossless objects can be spaced for zero reflection when and only when their transmission coefficients have the same absolute value. Hence the desired result is evident for any even number. For three objects A, B, C we know that A and B can be spaced to give zero reflection, and on the other hand they can be spaced to give reflection $2 \mid r \mid /(1 + \mid r \mid^2)$, if $\mid r \mid$ is the amplitude reflection coefficient of a single object. With $\mid R \mid$ as the reflection of the two, we accordingly have $\mid R \mid < \mid r \mid$ for one spacing and $> \mid r \mid$ for another; but $\mid R \mid$ is a continuous function of the spacing and must therefore assume the value $\mid r \mid$. Hence by the above the three can be spaced for zero reflection. This shows that any number can be spaced for zero reflection, since we have three or three plus an even number.

The spacing for the three objects can be uniform, for A and B together must have reflection $\mid r \mid$ (since the over-all reflection is zero); and B and C together must have reflection $\mid r \mid$ for the same reason.

To obtain the result for n objects assume it for $n - 1$ and use induction. The $n - 1$ objects can be equally spaced for zero reflection, and they can also be equally spaced for reflection $> \mid r \mid$ (see below). The result for n objects now follows by continuity as above. The proof shows that three objects can be spaced in two essentially distinct ways, and more than three in infinitely many ways.

4.3 Dissimilar lossless elements. If two lossless objects are spaced for maximum transmission, then (1) gives $\underline{/r_1} + \underline{/\rho} = 0$, and shows that the phase shift on reflection from the left is $\underline{/r}$, i.e., the same as it would be if the right-hand object were not present. Hence *the reflection from a lossless series is maximum when the elements are so spaced that the reflection of each adjacent pair is maximum.* If they are identical, in particular, then the reflection is maximum when they are equally spaced.

The foregoing is essentially a result on phase. For amplitude, it is convenient to introduce the standing-wave ratio s associated with an object, which is defined by

$$s = \frac{1 + \mid r \mid}{1 - \mid r \mid}. \tag{32}$$

Then *the standing-wave ratio for a series of lossless objects cannot exceed the product of the individual standing-wave ratios, and this value is always attained with some spacing.* The proof is easily given by induction if one notes that the maximum reflection in any series is an increasing function of the individual reflections.

It has been shown that $|t| = |\tau|$ for lossless objects, and hence that the reflection of a series is the same when the order of the objects is reversed. If the spacing may be changed there is a similar result for general permutations, not only for a reversal of order: *If the amplitude reflection for an array of lossless objects is equal to* $|R|$ *with the objects in any given order, then for any other order there exists a spacing such that the reflection is again equal to* $|R|$. For proof, note that any permutation can be attained by successive interchanges of adjacent objects. To keep the over-all reflection constant after such an interchange, we adjust the spacing so that the interchanged pair has the same amplitude reflection, then add a length of line on each side to give the same phase shift. The possibility of this procedure depends on the fact that $|t| = |\tau|$ for lossless objects.

A fundamental problem in series is to adjust the spacing for zero reflection, in which connection one has the following: Given a series of lossless objects, let $a, b, c, \cdots$ be their respective standing-wave ratios [see Eq. (32)]. Arrange the notation if necessary in such a way that $a \geq b, c, \cdots$. Then *the objects can be spaced to give zero reflection for the series when and only when* $a \leq bcd \cdots$. For proof, put the object with standing-wave ratio a at the extreme right, as may be done by the result just established. One can have zero reflection for the series when and only when the combined reflection of all objects except the last equals that of the last. One easily shows by induction that the minimum reflection of a series does not exceed the maximum reflection of the individual objects in the series. The first objects can be spaced, therefore, to give a standing-wave ratio $\leq a$; and also for one $\geq a$ if $a \leq bcd \cdots$, the maximum reflection possible (v. sup.). Hence by continuity they can be spaced to give a standing-wave ratio equal to a, and the result follows.

For two objects the minimum standing-wave ratio that can be attained is the quotient of the individual ones, the larger being the numerator. Also this minimum decreases as the smaller standing-wave ratio increases. From this, which is easily proved, we conclude in the above notation: *If* $a > bcd \ldots$, *then the minimum standing-wave ratio for the series is* $a/bcd \ldots$, *and this minimum is always attained for some spacing.*

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(Received March 2, 1949)

ITERATIVE COMPUTATION OF COMPLEX ROOTS

By P. A. Samuelson

1. In applied mathematics it is often necessary to evaluate complex roots of real polynomials, and in the study of oscillations it is frequently necessary to find the value of a polynomial for a number of complex arguments.[1] The present note points out that all this may be done by the same efficient methods as apply to the real variable problems—namely by synthetic division and by the classical methods of Newton, of "false position," and of generalized inverse interpolation. In addition there are set forth a few general considerations concerning iteration that are relevant to any comparison of the efficiencies of different methods that have been proposed. The recent method of Lin for finding complex roots is considered in some detail.

2. Let us consider a polynomial of the n^{th} degree with real coefficients

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = \sum_0^n a_i x^{n-i}$$

Suppose we wish the value of $f(x_1)$ where x_1 is real. In the literature one occasionally encounters cases where such an answer is arrived at by calculating powers of x and then adding. But these are rare since it is almost universally realized that the work may be reduced by a factor of n if instead we take advantage of the remainder-factor theorem, and calculate $f(x_1)$ by synthetic division. This involves about n successive multiplications and additions, since

$$f(x) = (x - x_1)[a_0'x^{n-1} + a_1'x^{n-2} + \cdots + a_{n-1}'] + f(x_1)$$

where

$$\begin{aligned} &a_0' = a_0,\ a_1' = x_1a_0' + a_1,\ a_2' = x_1a_1' + a_2 \cdots \\ &a_n' = x_1a_{n-1}' + a_n = f(x_1) \end{aligned} \tag{1}$$

and this process is ideally adapted to modern desk calculating machines.

If x_1 is complex, the same procedure is valid. But now we shall have to multiply complex numbers at each stage; and this, of course quadruples the work. As a result, De Moivre's theorem is often invoked, and recourse is made to trigonometric tables. Almost nothing is more tedious, and it would be better to accept a quadrupling of the work of synthetic division—if no better method were available.

3. But actually, by a simple extension of the most elementary polynomial identities, it is easy to halve the labor of calculating $f(x)$ for a complex x_1. Furthermore, only real quantities are employed up to the final step. The number of multiplications is still double that required in the real variable case; but as we shall see again and again, even this extra labor is mostly an illusion—since

[1] The first 5 references deal with the problem of complex roots, while the last reference gives a convenient discussion of all the usual numerical methods.

we simultaneously evaluate both $f(x_1)$ and $f(x_1^*)$ where x_1^* is the complex conjugate of x_1.
By repeatedly dividing our polynomial by terms of the form, $(x - x_1)$, $(x - x_2)$, $(x - x_3)$, $\cdots$, etc., we arrive at a simple extension of the remainder-factor theorem; namely

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_r)f(x, x_r, x_{r-1}, \cdots, x_1) + R \tag{2}$$

where $f(x, x_r, \cdots, x_1)$ is a polynomial of $(n - r)$ degree and where the remainder is a polynomial of $(r - 1)$ degree with the important property of having the same values as the original polynomial at each of the arguments $x = x_i$. That is,

$$f(x_i) = 0 + R(x_i, x_r, x_{r-1}, \cdots, x_1) \qquad (i = 1, 2, \cdots, r)$$

It may be noted that if s of the x_i's should be equal, then $(s - 1)$ of the derivatives of the remainder will have to equal the corresponding derivatives of the polynomial at such an x_i. It may also be remarked that the polynomial $f(x, x_r, \cdots, x_1)$ is the well-known Newtonian "divided difference of the r^{th} order" of our original polynomial, and is a symmetric function of its arguments. Should all of the arguments turn out to be equal then instantaneous derivatives will enter the picture in a manner familiar to all students of interpolation.

We may apply this analysis to the problem of evaluating a polynomial for a complex number, x_1. We consider the real quadratic expression $x^2 + b_1x + b_2 = (x - x_1)(x - x_1^*)$. We divide our polynomial by this quadratic expression, *encountering only real numbers in the process*, and arrive at a $(n - 2)$ degree quotient, $f(x, x_1, x_1^*)$, and a linear remainder; or

$$f(x) = (x^2 + b_1x + b_2)[a_0''x^{n-2} + \cdots + a_{n-2}''] + c_1''x + c_0''. \tag{3}$$

It follows that

$$f(x_1) = c_1''x_1 + c_0'' = (c_1'' \operatorname{Re} x_1 + c_0'') + i(\operatorname{Im} x_1c_1'') \tag{4}$$

For efficient computation one may employ the useful mnemonic device of algebraic long division, or the equivalent relations, derived from equating coefficients of like powers of x on both sides of (3):

$$\begin{aligned} a_0'' &= a_0 \\ a_1'' &= a_1 - b_1a_0'' \\ a_2'' &= a_2 - b_1a_1'' - b_2a_0'' \\ a_3'' &= a_3 - b_1a_2'' - b_2a_1'' \\ &\cdots\cdots\cdots\cdots \\ a_{n-2}'' &= a_{n-2} - b_1a_{n-3}'' - b_2a_{n-4}'' \end{aligned} \tag{5}$$

and

$$\begin{aligned} c_1'' &= a_{n-1} - b_1a_{n-2}'' - b_2a_{n-3}'' \\ c_0'' &= a_n - 0 - b_2a_{n-2}'' \end{aligned} \tag{6}$$

In all, these involve about $2n$ multiplications. This is twice that required for a real x, but $f(x_1^*)$ is computed along with $f(x_1)$.

As an illustration, one may consider the problem encountered in electrical and mechanical engineering of computing the complex impedance or transfer function of a differential equation with constant coefficients

$$f(d/dt)y(t) = e^{i\theta t}$$

whose steady-state solution is of the form

$$y(t) = [1/f(i\theta)]e^{i\theta t}$$

We calculate the remainder $c_1''x + c_0''$, of $f(x)/(x^2 + \theta^2)$, and find

$$f(i\theta) = c_0'' + i(c_1''\theta)$$

Here our complex number happens to be a pure imaginary and the work is halved, so that the example is rather a trivial one. But in the corresponding case of difference equations, met in many fields, where d/dt is replaced by the operator E, such that $Ey(t) = y(t + 1)$, we must evaluate terms of the form

$$y(t) = \frac{1}{f(\cos\theta + i\sin\theta)}[\cos\theta t + i\sin\theta t]$$

and here the efficiency of the present procedure is well illustrated.

4. Let us turn now to the problem of calculating complex or real roots of our polynomial. Two problems must always be distinguished: (1) that of determining *in the large* the general location of roots—by Graeffe's root-squaring method, Sturmian test functions, or general topological investigation of the transformation $y = f(x)$; and (2) improving the accuracy of a given root, or set of roots, once we have accomplished an approximate localization. It is primarily with this latter problem that most methods deal and with which this discussion will be primarily concerned.

The classical Newton method for calculating a real simple root is to use the iterative sequence (defined by replacing the function near a root by a tangential straight line)

$$x_{i+1} = H(x_i) = x_i - [f(x_i)/f'(x_i)] \tag{7}$$

It is easy to show that this converges to a simple root, A, for all initial values x_0 sufficiently near to A. Moreover, as is well-known, $H'(A) = 0$ and the convergence is exceedingly rapid, the number of "correct decimal places" increasing asymptotically in a geometric rather than an arithmetic progression. Such "quadratic or second-order convergence" is always sure of being eventually more accurate than any "linear" iteration for which $|H'(A)| \neq 0$, and whose correct decimal places eventually increase in an arithmetic progression. [6]

This does not mean that an efficient method is necessarily always to be preferred to a less efficient one. Except in the problem of "prodigious calculation"—such as evaluating $\sqrt{2}$ to a thousand decimal places—an asymptotically inefficient method may be satisfactory. But in no case should we pass over a

more efficient method unless there is some decided disadvantage in its application.

In the case of complex roots it is natural to consider using the same Newton method. In fact around the First World War, Edwin Bidwell Wilson used this method in aeronautical problems. But since then the method has come into disrepute, apparently for two quite different reasons. First, as we have seen, it is very time consuming to evaluate a polynomial for complex arguments by the conventional methods. The second objections do not relate peculiarly to the complex variable but are concerned with alleged inadequacies of Newton's method in the case of multiple roots or in the related case of closely clustered roots. Discussion of these second objections may be reserved for a later section.

By using the method just outlined, we may expeditiously evaluate $f(x_i)$ and $f'(x_i) = \sum_1^n a_{n-j} j x^{j-1}$. The work is double that for a real variable, but we approximate to two conjugate roots A, and A^*. In practice, even in the real variable case we need not work directly with the polynomial, $f'(x)$, but can calculate $f'(x_i)$ by a second synthetic division of $P(x)$ by $(x - x_i)$, the remainder c_1'' being the required derivative. A similar artifice can be followed in the complex variable case. For,

$$
\begin{aligned}
f(x) &= (x - x_i)f(x, x_i) + f(x_i) \\
&= (x - x_i)(x - x_i^*)f(x, x_i^*, x_i) + (x - x_i)\, f(x_i^*, x_i) + f(x_i) \\
f'(x_i) &= 0 + (x_i - x_i^*)f(x_i, x_i^*, x_i) + f(x_i, x_i^*) + 0 \\
&= (x_i - x_i^*)(c_2'''' - x_i^* c_3'''') + c_1''
\end{aligned} \tag{8}
$$

Our only task is to evaluate $f(x, x_i^*, x_i) = \sum_0^{n-2} a_j'' x^{n-j}$ for $x = x_i$. This too can be done by the method indicated for $f(x)$ itself; just as we can get (a_i'') from (a_i), so we can get (a_i'''') from (a_i''). In short, by twice dividing $f(x)$ by $x^2 - (x_i + x_i^*)x + x_i x_i^*$, we are in a position to evaluate $f(x_i)$, $f'(x_i)$, $f(x_i^*)$, $f'(x_i^*)$. By more divisions by the same quadratic factor we can calculate higher derivatives. But since each such division reduces the degree of our polynomial by 2, we can apparently only hope to evaluate about half the non-vanishing higher derivatives of our polynomial by such methods.

Just as all parts of the proof of the convergence of the Newton method hold as well for complex as for real variables, so too the proof holds for the convergence of the even more ancient "rule of false position" method. In this method we replace the function by the straight line secant joining two of its points in the neighborhood of a root, and solve for a third better point. Thus, we have defined a difference equation of the second order.

$$
x_{i+2} = H(x_{i+1}, x_i) = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)} \tag{9}
$$

This is nothing but simple linear inverse interpolation. It is easy to verify by differentiation that in the neighborhood of any simple root, A, we have

$$
\frac{\partial H}{\partial x_{i+1}}(A, A) = 0 = \frac{\partial H}{\partial x_i}(A, A) \tag{10}
$$

so that the process must converge for all initial values x_0, x_1 sufficiently close to A. Moreover, the convergence is very rapid, again being asymptotically quadratic rather than linear.

As usually expounded in the text books, the rule of false position is used to interpolate between two values of $f(x)$ of opposite sign. This is unnecessary, and in fact undesirable: following the above iteration will lead to convergence to the true root by *extra*polation, a more efficient computational procedure.

Correctly applied, Newton's method and the rule of false position are about equally powerful and involve about the same amount of work. They are both much more efficient than the usually-taught Horner's method which proceeds one digit at a time, and which in order to diminish the roots of the polynomial by the appropriate numbers requires that *all* the derivatives of the function be calculated at *each* stage. Even with synthetic division, this involves a good deal of unnecessary work, especially after the very first stages. This is perceived in the more advanced treatments of the method (viz. Burnside and Panton's classical discussion); and it stems from the fact that Newton's method and the Rule of False Position quickly outrun single digit at a time accuracy, so that the practitioner of Horner's method soon feels that he is almost marking time.

The following numerical example is designed to bring into immediate play the relations which always hold after a number of whirls of any of the methods under discussion. The equation

$$f(x) = .001x^2 - x - .01 = 0$$

clearly has roots in the neighborhood of 1,000 and $-.01$. The correct larger root to 10 significant figures is 1,000.0099999 which could be determined by Newton's method with 3 whirls, or only a fraction of the work of the usual full Horner procedure.

5. The prevailing distrust of Newton's method, which dates back at least to Lagrange, seems to stem from its alleged inadequacies in the case of multiple roots. In such cases the denominator, $f'(x_i)$ is a very small number and the correction at each stage is the ratio of two very small numbers. Nonetheless, it can be shown that Newton's method does converge to a multiple root for all sufficiently close initial values; but the convergence is now slower, being linear rather than quadratic. In fact

$$0 \neq H'(A) = (s-1)/s \tag{11}$$

where s is the multiplicity of the root, A.

If we knew in advance that the sought for root was of multiplicity, s, we could solve for the root of $f^{(s-1)}(x) = 0$ by the efficient iteration

$$x_{i+1} = x_i - [f^{(s-1)}(A)/f^{(s)}(A)] \tag{12}$$

But in numerical work, perfectly equal roots only occur as the result of some theoretical idealization of the empirical situation. We rarely can be sure in advance that the sought for root is multiple, so that the above artifice is not very useful.

In applied numerical work, we not infrequently do encounter the case of roots which cluster very close together. And it might be argued that this, rather than multiplicity of roots, provides a challenge to Newton's method, because in some near neighborhood of the root, the first derivative will vanish and cause the iteration to blow up. Consider as an illustration the simple examples $x^3 = 0$, and $x^3 = 1$. Newton's iteration, in the first case being $x_{t+1} = 2/3x_t$, has a domain of convergence of the whole complex plane. Here root multiplicity actually somewhat simplifies the problem of convergence in the large. But in the second example, $x = 0$ is a singular point of the iteration.

The difficulties arising from close or coincident roots are intrinsic. They show up in other methods than Newton's, both in locally efficient and inefficient procedures. Root squaring or other transformations cannot alter the basic qualitative topology of the problem in the large. Since there are neighborhoods of convergence around each root, there must always be watersheds to mark the no-man's frontier between each root's domain. What root squaring and other transformations can do is to help us to land in the convergence neighborhood of a particular root.

From a physical point of view we cannot always regard multiple or clustered roots as a nuisance; they may even be a positive advantage. Thus, if our problem is to locate a relative extremum (or stationary point) of $\int f(x)dx$, a multiple root means that the extremum value is nearly constant in a wide region near to the exact root, $x = A$. In still other applications, our concern is with the degree of closeness to which $|f(x)| = 0$, rather than $|x - A| = 0$, and the sequence $f(x_t)$ converges no less rapidly because of multiple roots; in fact, if one can use (12) or any other artifice to get x_t to converge with quadratic order to a multiple root, then $f(x_t)$ will converge with still higher order rapidity.

Often, too, where nature has clustered roots in a close array, it serves no useful purpose for man to try to unscramble them. For most practical purposes we may treat close roots as being multiple ones, especially since the constants in any empirical equation are subject to error. When the roots are close together, the separate identity of each becomes sensitive to slight changes in the data. This should not be taken to mean that the mathematical features of the total phenomena are "ill-conditioned"; it may be only the representation into particular components that is ambiguous.

6. Thus far, no explicit attention has been given to methods which approach a root with only linear convergence. Bernoulli's method of approximating to the largest root of a polynomial by means of a difference equation sequence is such a method. So too is the closely related Whittaker series method [7]. A popular variant of the Newton iteration is the simpler sequence, $x_{t+1} = x_t - f(x_t)/f'(B)$, where $f'(B)$ is the derivative evaluated at some fixed point near-by to the root, A. This is a special case of the sequence

$$x_{t+1} = x_t - [f(x_t)/R(x_t)] \tag{13}$$

where R is some suitably defined function: e.g. R might be the slope of a secant. When $R(x) = f'(x)$, we have the Newton method; when $R(x)$ is a constant, we

have the just mentioned variant of the Newton method. For local convergence, it is sufficient that

$$|1 - [f'(A)/R(A)]| < 1 \tag{14}$$

But, unless $R(A) = f'(A)$, the convergence will be at best linear, and if A is a multiple root corresponding to an extremum of $f(x)$, then with $R \neq 0$ the process will not be convergent even for initial approximations arbitrarily close to A. Thus, when $R(x) \equiv f(b) \neq f'(A) = 0 \neq f''(A)$, there will be divergence of the sequence for all initial values on one side of A. However, for simple roots this sequence may behave almost as well during the first few iterations as an asymptotically more efficient method.

7. Lin [3, 4] has proposed a general method for evaluating a pair of real or complex roots. This method can be used to approximate a single real (or complex) root; or to approximate simultaneously several pairs of complex roots. There are conditions under which the method can be used to advantage; but nonetheless the method seems to lack two of the desirable properties of Newton's Method or the Method of False Position. First, it need not converge to a simple (or multiple) root even if the initial guesses are infinitesimally close to the correct values. Second, its convergence is at best linear rather than quadratic. There seem to be further difficulties inherent in the method when the roots of the polynomial happen to fall into special patterns and constellations. Also, rigorous conditions for convergence can be defined only in rather complicated terms, although simpler heuristic criteria may be suggested.

The essence of the Lin method is to set the right hand members of equation (6) to zero, and solve for a new, improved set of b's. In effect, the Lin method makes no use of the values of the remainder terms resulting from previous approximations to a root—since the method assumes the remainders are all zero. But, as the Newton method shows, it is the quantitative values of the remainder that permit the most precise estimate of the needed correction to the true root, so that a valuable part of our available information is being discarded.[2]

This may best be illustrated by examining the Lin method as applied to a single root. Given an approximate value, x_t, we seek a better approximation, x_{t+1}. Dividing $f(x)$ by $(x - x_t)$ gives

$$f(x) = (x - x_t)[a'_0 x^{n-1} + a'_1 x^{n-1} + \cdots + a'_{n-1}] + f(x_t)$$

where

$$f(x_t) = f(0) + x_t a'_{n-1}$$

Hence, rearranging terms and utilizing (6) according to Lin's technique, we find

$$x_{t+1} = \frac{f(0)}{a'_{n-1}} = \frac{x_t a'_{n-1} - f(x_t)}{a'_{n-1}} = x_t - \frac{f(x_t)}{a'_{n-1}} = x_t - \frac{f(x_t)}{\dfrac{f(x_t) - f(0)}{x_t - 0}} \tag{15}$$

[2] The Hitchcock method [5] recognizes this fact; like Newton's method it involves two divisions of the polynomial by a trial quadratic factor, and correction of the quadratic factor by means of the remainder. Its convergence seems to be of a slower order than Newton's.

where

$$\frac{f(x_t) - f(0)}{x_t - 0} = \left(\frac{\Delta f}{\Delta x}\right)_{0,x_t} = R(x_t)$$

This has a simple graphical interpretation. It differs from the Newton method only in the denominator of the last term. Instead of drawing a line tangent to the curve through the point $[x_t, f(x_t)]$, we draw a line through that point with slope equal to that of the secant between $[0, f(0)]$ and $[x_t, f(x_t)]$. If the desired root A is small in absolute value, then this will not differ much from the Newton process or the Rule of False Position. The error, $(x_t - A)$ will ultimately behave like

$$(x_{t+1} - A) = [1 - \{f'(A)/R(A)\}](x_t - A) + \cdots$$

or nearly proportional to $[1 - \{f'(A)/R(A)\}]^t$.

For the Lin process to converge, it is not enough for $| x_0 - A |$ to be sufficiently small. In addition the equation must have been transformed so that $| A |$ is "small". Thus, if $f(x) = x^2 + a_1x + a_2$, the Lin process will only converge to that root which is smallest in absolute value.

It would seem that in the more elaborate case where two roots are sought, convergence depends upon these two roots being "small"—not absolutely but in comparison with all other roots. Thus consider the cubic

$$\begin{aligned} x_3 + a_1x^2 + a_2x + a_3 &= (x^2 + B_1x + B_2)(x + K) \\ &= x^3 + (B_1 + K)x^2 + (B_1K + B_2)x + KB_2 \end{aligned} \tag{16}$$

where the coefficients may be real or complex. Suppose we guess an initial set of values $b_1(0)$, $b_2(0)$, which are near to B_1 and B_2. For what values of (B_1, B_2, K) will the resulting difference equation be convergent in a neighborhood of (B_1, B_2)?

Our iterative sequence for $b_1(t + 1)$, $b_2(t + 1)$ in terms of $b_1(t)$, $b_2(t)$ is from (16), (5), and (6) defined by

$$\begin{aligned} b_2(t + 1) &= \frac{a_3}{a_1} = \frac{KB_2}{B_1 + K - b(t)} \\ b_1(t + 1) &= \frac{a_2 - b_2a_0''}{a_1''} = \frac{(B_2 + KB_1) - [KB_2/\{B_1 + K - b_1(t)\}]}{B_1 + K - b_1(t)} \end{aligned} \tag{17}$$

The second of these relations is a difference equation for $b_1(t)$ alone. Regardless of K, it has a stationary point for $b_1(t) = B_1$, and it will converge to B_1 for all initial values $b_1(0)$ sufficiently near to B_1, if at the point $b_1 = B_1$

$$\left|\frac{\partial b_1(t + 1)}{\partial b_1(t)}\right| = \left|\frac{B_2}{K_2} - \frac{B_1}{K}\right| < 1 \tag{18}$$

The Lin process must therefore be divergent wherever the absolute value of the third root is small relative to the sum and products of the pair of roots sought. Doubtless a similar qualitative relation holds in the general case. Con-

sequently it is only after certain preliminary transformations on the roots of the polynomial have been performed, that the method will enable us to locate certain specific roots; or only after other roots have been found, and have been used to reduce the degree of the equation.

7. In conclusion, I should like to point out that no mention has been made of the problem of rounding off errors. All of the enunciated theorems concerning convergence apply to the exact iterative sequence. In practice we must always round off, and even the most powerful method can be rendered nugatory by the manner in which we do so. In fact, the really powerful methods, as is well known, secure their power by squeezing the last bit of information out of every digit; there is therefore, always the danger that in rounding off we may throw out the baby along with the bath water, so as to end up with 0/0, or what is worse, with the ratio of one small random error to another such error. Rounding off errors can be thought of as "noise" impinging on our dynamic sequences. Much remains to be done in analyzing the effect of such shocks on different kinds of processes. The whole subject—of degree of convergence and of rounding error—acquires a new significance in the modern era of giant calculating machines.

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(Received December 15, 1948)

THE LINEAR-PERTURBATION THEORY FOR ROTATIONAL FLOW

By W. R. Sears

1. In a current investigation,[1] the steady flow conditions behind a "normal" shock wave slightly disturbed by an inhomogeneity of the supersonic stream are being studied. It is desirable in this case, and perhaps in other cases, to employ the ideas of the so-called linear-perturbation theory in a region characterized by small, but non-vanishing, vorticity. The equations governing this type of flow will be set up here. It appears that they may be generally useful in problems involving small-perburbation type of flow with strong shock waves, as well as in problems of compressible shear flow.

2. Consider plane or axisymmetric steady flow with velocity components u and v along the x and y directions. Suppose U is a constant such that

$$u = U + u' \quad \text{and} \quad v = v' \tag{1}$$

where $$u' \ll U \qquad \text{and} \quad v' \ll U.$$

Suppose, furthermore, that the stagnation enthalpy H is constant; i.e.,

$$\tfrac{1}{2}q^2 + [\gamma/(\gamma - 1)](p/\rho) = H = \text{const.} \tag{2}$$

where $q^2 = u^2 + v^2$, γ denotes the ratio of specific heats, p the pressure, and ρ the density. This condition is satisfied whenever the flow proceeds from a region of uniform flow isentropically except for the presence of shock waves. Let the subscript $(\ )_1$ refer to conditions in a parallel stream of speed U, such that

$$\tfrac{1}{2}U^2 + [\gamma/(\gamma - 1)]\,(p_1/\rho_1) = H. \tag{3}$$

Then, neglecting second-order terms,

$$p/\rho = p_1/\rho_1 - [(\gamma - 1)/\gamma]Uu'. \tag{4}$$

Naturally, the parallel stream characterized by U, p_1, and ρ_1 may be fictitious; nevertheless these are convenient reference parameters, and are completely defined, for given H, by eqs. (1)–(3).

Eq. (4) can be converted into a formula for the density ratio by use of the equation of state for a perfect gas,

$$p/p_1 = (\rho/\rho_1)^\gamma \exp\,[(S - S_1)/c_v] \tag{5}$$

in which S denotes entropy and c_v the specific heat at constant volume. Eq. (4) becomes

$$\begin{aligned} \frac{\rho}{\rho_1} &= \left\{1 - (\gamma - 1)M\,\frac{u'}{a_1}\right\}^{1/(\gamma-1)} \exp\left(\frac{-1}{\gamma - 1}\,\frac{S - S_1}{c_v}\right) \\ &\approx \left(1 - M\,\frac{u'}{a_1}\right) \exp\left(\frac{-1}{\gamma - 1}\,\frac{S - S_1}{c_v}\right) \end{aligned} \tag{6}$$

where M denotes U/a_1 and a_1 the velocity of sound, $(\gamma p_1/\rho_1)^{1/2}$.

[1] M. C. Adams: On Shock Waves in Inhomogeneous Flow. J. Ae. Sc. **16**, 685–690 (1949).

The entropy S_1 is, of course, constant, since it occurs in a uniform parallel stream wherein H is constant; furthermore, S is related to the vorticity Ω by Bjerknes' theorem:

$$v_x - u_y = \Omega = (p/\rho q R)\,|\,\text{grad}\,S\,| \tag{7}$$

where R is the gas constant $c_p - c_v$.

It is clear from eq. (7) that the vorticity and grad S, and therefore also $S - S_1$, are small quantities, at most of first order. Consequently eq. (6) can be written in a more consistent approximate form; viz.,

$$\frac{\rho}{\rho_1} = \left(1 - M\frac{u'}{a_1}\right)\left(1 - \frac{1}{\gamma - 1}\frac{\delta S}{c_v}\right) \approx 1 - M\frac{u'}{a_1} - \frac{\delta S}{R}\,. \tag{8}$$

Here we have written δS for $S - S_1$, and neglected second-order terms throughout.

3. Let us now introduce the appropriate stream function ψ, defined by

$$(\rho/\rho_1)uy^\epsilon = \psi_y\,, \qquad (\rho/\rho_1)vy^\epsilon = -\psi_x \tag{9}$$

where ϵ is zero for plane flow and unity for axi-symmetric. Moreover, let us write

$$\psi = U(1 + \epsilon)^{-1}y^{1+\epsilon} + \psi' \tag{10}$$

so that ψ' is a perturbation stream function.[2] Then, by virtue of eq. (8), we have

$$\left.\begin{aligned}\left(1 - M\frac{u'}{a_1} - \frac{\delta S}{R}\right)(U + u')y^\epsilon &= Uy^\epsilon + \psi'_y\\ \left(1 - M\frac{u'}{a_1} - \frac{\delta S}{R}\right)v'y^\epsilon &= -\psi'_x\end{aligned}\right\} \tag{11}$$

or, to the first order,

$$(1 - M^2)u' - U(\delta S/R) = \psi'_y y^{-\epsilon}, \quad v' = -\psi'_x y^{-\epsilon} \tag{12}$$

It remains to express $\delta S/R$ in terms of the velocity components, and this is easily done with the aid of eq. (7); we have, for this type of flow,

$$|\,\text{grad}\,S\,| = (dS/d\psi)\,(\rho/\rho_1)y^\epsilon q$$

and therefore

$$\Omega = (p/\rho_1 R)y^\epsilon(dS/d\psi)$$

Thus

$$\delta S/R = \rho_1 \int (v_x - u_y)y^{-\epsilon}\,d\psi/p = \int (v'_x - u'_y)\,(u\,dy - v\,dx)\rho/p. \tag{13}$$

[2] Cf. S. Goldstein and A. D. Young: Aero. Res. Comm. Reports and Memoranda No. 1909 (1943), page 2.

But it is certainly permissible to eliminate second-order terms in this expression and therefore we write

$$\delta S/R = U(\rho_1/p_1)\int (v_x' - u_y')\,dy\,. \tag{14}$$

This is the formula to be used in eqs. (12).

The differential equation for ψ' is now obtained by solving eqs. (12) for u_y' and v_x', and calculating the vorticity:

$$(1 - M^2)u_y' - \gamma M^2(v_x' - u_y') = \psi_{yy}' y^{-\epsilon} - \epsilon\psi_y' y^{-\epsilon-1}$$

$$v_x' = -\psi_{xx}' y^{-\epsilon}$$

from which

$$u_y' = \frac{\psi_{yy}' - \epsilon y^{-1}\psi_y' - \gamma M^2\psi_{xx}'}{1 + (\gamma - 1)M^2}\, y^{-\epsilon}$$

and finally

$$(1 - M^2)\psi_{xx}' + \psi_{yy}' - \frac{\epsilon}{y}\psi_y' = -\{1 + (\gamma - 1)M^2\}y^{\epsilon}\Omega\,. \tag{15}$$

Eq. (15) is to be satisfied by the perturbation stream function in rotational flow. In application of this equation it will be consistent with our first-order theory, to consider the vorticity Ω to be a function of y only.

It might be mentioned that, except in the small-perturbation case considered here, the conventional stream function defined in eqs. (9) is not a particularly useful device in rotational compressible flow problems. This has led Crocco[3] to define a different stream function, which differs from the usual one by a factor involving the entropy. Crocco worked out the differential equation satisfied by his new function, and it has been verified that eq. (15) is consistent with it, in the perturbation case.

4. In subsonic-flow problems ($M < 1$), eq. (15) can be transformed into equation of incompressible flow, by application of the familiar Prandtl-Glauert affine transformation:

$$\xi = x, \qquad \eta = \sqrt{1 - M^2}\, y = \beta y, \qquad \text{say} \tag{16}$$

Thus, for plane flow, it becomes Poisson's equation in the ξ, η plane.

It may be of interest to relate the compressible rotational flow past a given body to a corresponding incompressible flow. Suppose the following incompressible-flow problem has been solved: A stream of speed μU flows rotationally past a body (cylindrical or axisymmetric) defined by the profile curve $\eta = H(\xi)$,

[3] L. Crocco: Zeits. f. angew. Math. u. Mech. 17, 1–7 (1937).

the vorticity being given by the function $F(\xi, \eta)$; that is,

$$\left.\begin{aligned} &\Psi = \mu U(1+\epsilon)^{-1}\eta^{1+\epsilon} + f(\xi, \eta) \\ &f_{\xi\xi} + f_{\eta\eta} - (\epsilon/\eta)f_\eta = -\eta^\epsilon F(\xi, \eta) \\ &\Psi = 0 \text{ defines } \eta = \mathrm{H}(\xi); \text{ i.e.} \\ &\mu U(1+\epsilon)^{-1}H(\xi)^{1+\epsilon} + f[\xi, H(\xi)] = 0 \end{aligned}\right\}. \tag{17}$$

Then, by virtue of eqs. (16), $\psi' = f(x, \beta y)$ is a solution of eq. (15), provided that

$$[1 + (\gamma - 1)M^2]\beta^{-\epsilon}\Omega(x, y) = F(x, \beta y). \tag{18}$$

Thus eq. (18) is the formula that prescribes the vorticity distribution $F(\xi, \eta)$ in the associated incompressible flow. The compressible stream function can now be written as

$$\psi = U(1+\epsilon)^{-1}y^{1+\epsilon} + f(x, \beta y) \tag{19}$$

and the streamline $\psi = 0$ yields the profile

$$\beta y = H(x) \tag{20}$$

provided that

$$U(1+\epsilon)^{-1}y^{1+\epsilon} = \mu U(1+\epsilon)^{-1}H(x)^{1+\epsilon}$$

or

$$y = \mu^{1/(1+\epsilon)}H(x) \tag{21}$$

Eqs. (20) and (21) require that

$$\mu = \beta^{-(1+\epsilon)}. \tag{22}$$

In summary, then, the compressible steady flow with vorticity Ω, past the body $y = Y(x)$, is related to an incompressible flow of reduced stream speed, with vorticity F, past a thinner body $\eta = \beta Y(\xi)$. Except for the new information about vorticity, the rule is precisely the same as in irrotational flow.[4, 5]

The force on the body immersed in such a compressible flow can be related to the force that acts in the corresponding incompressible case. It is found that the rule relating the forces in the incompressible and the associated incompressible flows is the same as for irrotational flow. Moreover, it appears from a study of this question that these forces will not be affected by the vorticity, to the first order, nor will there appear a first-order drag.

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(Received March 25, 1949)

[4] B. Göthert: N.A.C.A. Tech. Memo. 1105 (1946).
[5] W. R. Sears: Quart. Appl. Math. 5, 89–91 (1947).

SUMMATION OF SLOWLY CONVERGENT SERIES*

By Otto Szász[1]

1.1. Most of the classical summability methods assign to a series $\sum u_\nu$ (convergent or not) the "sum" $\lim y_n$, if this exists, where

$$y_n = \sum_{\nu=0}^{n} a_{n\nu} x_\nu , \tag{1.1}$$

and

$x_n = \sum_0^n u_\nu$ are the partial sums of a series $\sum_0^\infty u_\nu$; $(a_{n\nu})$ is a triangular matrix with positive elements $a_{n\nu} \geq 0$, and we assume $\sum_{\nu=0}^{n} a_{n\nu} = 1, n = 0, 1, 2, \cdots$.

If such a method sums every convergent series to its proper value it is said to be regular. This is the case for the transform (1.1) if

$$\lim_{n\to\infty} a_{n\nu} = 0, \quad \text{for} \quad \nu = 0, 1, 2, \cdots .$$

1.2. *Examples.* 1. *The method of arithmetic means*:

$$y_n = \frac{1}{n+1}\sum_0^n x_\nu = \frac{1}{n+1}\sum_0^n (n+1-\nu)u_\nu . \tag{1.2}$$

2. *Cesàro means of order* α:

$$y_n = \frac{1}{\gamma_n^{(\alpha)}}\sum_0^n x_\nu \gamma_{n-\nu}^{(\alpha-1)} = \frac{1}{\gamma_n^{(\alpha)}}\sum_0^n u_\nu \gamma_{n-\nu}^{(\alpha)} ,$$

where

$$\gamma_n^{(\alpha)} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} .$$

$\alpha = 1$ reduces to example 1.

3. *Nörlund means*:

$$y_n = \frac{1}{\lambda_n}\sum_0^n p_{n-\nu} x_\nu , \qquad \lambda_n = \sum_0^n p_\nu , \qquad p_n \geq 0, \qquad \frac{p_n}{\lambda_n} \to 0. \tag{1.3}$$

$p_n = \gamma_n^{(\alpha-1)}$ reduces to example 2.

4. *Typical means*:

$$y_n = (1/\lambda_n)\sum_0^n p_\nu x_\nu , \; p_n \geq 0, \; \lambda_n = \sum_0^n p_\nu \to \infty . \tag{1.4}$$

5. $$y_n = (1-a_n)x_n + a_n x_{n-1} , \; 0 \leq a_n \leq 1.$$

* Presented to the American Mathematical Society June 19, 1948

[1] This paper was written at the Institute for Numerical Analysis of the National Bureau of Standards, with the financial support of the Office of Naval Research of the U. S. Navy Department.

6. *Hausdorff means*:

$$a_{nv} = \binom{n}{v} \int_0^1 t^v (1-t)^{n-v} q(t)\, dt, \qquad \int_0^1 q(t)\, dt = 1,\ q(t) \geq 0.$$

$q(t) = \alpha(1-t)^{\alpha-1}$ reduces to example 2.

7. *Euler means*:

$$a_{nv} = \binom{n}{v} t^v (1-t)^{n-v}, \qquad 0 \leq t < 1.$$

1.3. While such transforms sum many divergent, oscillating series, they converge slower than the sequence $\{x_n\}$, when $\{x_n\}$ is an increasing sequence, that is $u_n > 0$. For then

$$y_n < x_n \textstyle\sum_0^n a_{nv} = x_n < s, \text{ where } x_n \uparrow s.^2$$

In other words if $x_n = \sum_{v=0}^n b_{nv} y_v$ is the inverse of the transform (1.1), then under the given assumption x_n is closer to s than y_n.

This suggests the use, for series with positive terms, of nonregular transforms or the inverse of a regular transform. We discuss in detail the effect of transformations inverse to transformations by typical means.

2.1. From (1.4)

$$p_n x_n = \lambda_n y_n - \lambda_{n-1} y_{n-1},\ n \geq 1,\ x_0 = y_0,\ p_n = \lambda_n - \lambda_{n-1},$$

so that

$$x_n = \frac{1}{p_n}\left\{(\lambda_n - \lambda_{n-1}) y_n + \lambda_{n-1}(y_n - y_{n-1})\right\} = y_n + \frac{\lambda_{n-1}}{p_n}(y_n - y_{n-1})$$

is the inverse of (1.4). We shall apply this type of transform to a certain class of series.

We assume that $u_n > 0$, and that for some constant $a > 1$

$$\frac{u_n}{u_{n-1}} = 1 - \frac{a}{n} + o\left(\frac{1}{n}\right).$$

We write

$$\frac{u_n}{u_{n-1}} = 1 - \frac{a}{n} + \frac{\gamma_{n-1}}{n}, \qquad \gamma_n \to 0. \tag{2.1}$$

Hence

$$u_n = u_{n-1} - \frac{a}{n} u_{n-1} + \frac{\gamma_{n-1}}{n} u_{n-1}, \tag{2.2}$$

[2] In a stricter sense x_n converges slower to x than y_n to y, or more rapidly, according as $\left|\frac{x_n - x}{y_n - y}\right| \to \infty$ or $\frac{x_n - x}{y_n - y} \to 0$.

and

$$nu_n = (n-1)u_{n-1} + (1-a)u_{n-1} + \gamma_{n-1}u_{n-1}. \tag{2.3}$$

It follows from (2.1) that $n^a u_n = 0(1)$, hence $nu_n \to 0$ and $\sum |u_n|$ is convergent; from (2.3), replacing n by $v+1$

$$vu_v - (v+1)u_{v+1} = (a-1)u_v - \gamma_v u_v,$$

and, in view of $nu_n \to 0$,

$$nu_n = (a-1)\sum_n^\infty u_v - \sum_n^\infty \gamma_v u_v. \tag{2.4}$$

Hence, if $\sum_0^n u_v = s_n \to s$,

$$(a-1)s_{n-1} + nu_n = (a-1)s - \sum_n^\infty \gamma_v u_v. \tag{2.5}$$

Let

$$\max_{v \geq n} |\gamma_v| = \bar{\gamma}_n, \quad \bar{\gamma}_n \downarrow 0,^3$$

then (2.5) yields

$$|(a-1)(s-s_n) - (n+1)u_{n+1}| < \bar{\gamma}_{n+1}\sum_{n+1}^\infty u_v = \bar{\gamma}_{n+1}(s-s_n)$$

or

$$\left| s - s_n - \frac{(n+1)u_{n+1}}{a-1} \right| < \frac{\bar{\gamma}_{n+1}}{a-1}(s-s_n).$$

We have now established the following result:

Theorem 1. *If* (2.1) *holds then the transform* $t_n = s_n + \dfrac{(n+1)u_{n+1}}{a-1}$ *converges more rapidly than* s_n ; *we have*

$$|s - t_n| < \frac{\bar{\gamma}_{n+1}}{a-1}(s-s_n), \quad \textit{where} \quad \bar{\gamma}_n = \max_{v \geq n} |\gamma_v| \downarrow 0. \tag{2.6}$$

The transformation t_n used in this theorem can be iterated, provided stronger assumptions are made on u_n. The details of the examination are omitted.

2.2. The case $u_n = (n+1)^{-2}$ (hence $a = 2$, $s = \pi^2/6$) has been studied numerically[4]; the results obtained[5] suggested that the transform yields a decreasing sequence. That this is the case for a general class of series is shown by our next theorem.

Theorem 2. *Assume that* $u_n > 0$, *and that for some constants* $\alpha > 1$, $\beta > -1$

$$u_{n+1} \leq \frac{n+\beta}{n+\alpha+\beta} u_n, \qquad n = 1, 2, 3, \cdots; \tag{2.7}$$

[3] The symbol $\downarrow$ means that the sequence is monotonic decreasing.

[4] W. G. Bickley and T. C. B. Miller, The numerical summation of slowly convergent series of positive terms, Philosophical Magazine, Ser. 7, vol. 22, 1936, pp. 754–767.

[5] loc. cit. p. 760.

then

$$s_n + \frac{n+\beta}{\alpha-1}u_n \downarrow s.$$

It follows from our assumption (2.7) that $u_n = O(n^{-\alpha})$, so that $\sum u_n$ converges, and

$$s_n' = s_n + \frac{n+\beta}{\alpha-1}u_n - s.$$

Furthermore

$$s_n' - s_{n+1}' = s_n - s_{n+1} + \frac{n+\beta}{\alpha-1}u_n - \frac{n+\beta+1}{\alpha-1}u_{n+1}$$

$$= \frac{n+\beta}{\alpha-1}u_n - \frac{n+\beta+1+\alpha-1}{\alpha-1}u_{n+1} = \frac{n+\beta}{\alpha-1}u_n - \frac{n+\beta+\alpha}{\alpha-1}u_{n+1},$$

and this is non-negative by assumption (2.7). This proves Theorem 2.
In particular for $u_n = (n+1)^{-2}$ and $\alpha = 2$, the assumption becomes

$$\frac{1}{(n+2)^2} \leq \frac{n+\beta}{n+2+\beta}\cdot\frac{1}{(n+1)^2},$$

or

$$(n^2+2n+1)(n+2+\beta) \leq (n+\beta)(n^2+4n+4),$$

that is

$$2n^2 + (2+\beta)n^2 + n + 2(2+\beta)n + 2 + \beta \leq (\beta+4)n^2 + 4\beta n + 4n + 4\beta,$$

or

$$(2\beta+5)n + 2 + \beta \leq (4\beta+4)n + 4\beta,\ 2 - 3\beta \leq (2\beta-1)n,\ n = 1, 2, 3, \cdots$$

Hence, if $2 - 3\beta \leq 2\beta - 1$, or $3 \leq 5\beta$, $\beta \geq 3/5$, then our assumption is satisfied.
Example

$$\sum_0^\infty u_n = \sum_0^\infty \frac{1}{(4n+1)(4n+3)} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots = \frac{\pi}{8}.$$

Now $\pi/8 = 0.39270\cdots$,

$$s_0 = 0.33333\cdots, \qquad s_1 = 0.36190\cdots,$$
$$s_2 = 0.37200\cdots, \qquad s_3 = 0.37712\cdots,$$
$$s_4 = 0.38021\cdots, \qquad s_5 = 0.38228\cdots,$$
$$s_6 = 0.38376\cdots, \qquad s_7 = 0.38487\cdots.$$

On the other hand in Theorem 1 $\alpha = 2$, $t_n = s_n + (n+1)u_{n+1}$;

$$t_0 = s_1, \quad t_1 = 0.38210\cdots, \quad t_2 = 0.38127\cdots,$$
$$t_3 = 0.38540\cdots, \quad t_4 = 0.38761\cdots,$$
$$t_5 = 0.38894\cdots, \quad t_6 = 0.38981\cdots,$$
$$t_7 = 0.39041\cdots.$$

Thus the approximation to $\pi/8$ by t_n is considerably faster than by s_n. Furthermore Theorem 2 holds with $\alpha = 2, \beta = 1$, so that $s_n + (n+1)u_n = t_{n-1} + 2u_n$ is decreasing to $\pi/8$.

2.3. The essence of our procedure is to compare in the case of a convergent series $\sum_0^\infty u_n$, where $u_n \downarrow$, nu_n and $\sum_{n+1}^\infty u_\nu = r_n$.

It is known that in this case $nu_n \to 0$; in fact

$$u_{n+1} + \cdots + u_{2n} > nu_{2n} > nu_{2n+1}, \text{ or}$$

$$nu_{2n+1} < nu_{2n} < r_n - r_{2n} \to 0, \tag{2.8}$$

hence $nu_n \to 0$.

An improvement of this relation may be of interest. We have

$$u_{n+1} + \cdots + u_{2n} < nu_n,$$

or

$$r_n - r_{2n} < nu_n = n(r_{n-1} - r_n). \tag{2.9}$$

Hence

$$(n+1)r_n < nr_{n-1} + r_{2n},$$

so that

$$\sum_1^n (\nu+1)r_\nu < \sum_1^n \nu r_{\nu-1} + \sum_1^n r_{2\nu}.$$

From (2.8) and (2.9)

$$nu_{2n} < r_n - r_{2n} < nu_n;$$

hence, if we write $u(n)$ for u_n and $r(n)$ for r_n

$$n2^\nu u(n2^{\nu+1}) < r(n2^\nu) - r(n2^{\nu+1}) < n2^\nu u(n2^\nu), \nu = 0, 1, 2, \cdots,$$

and

$$n\sum_{\nu=0}^\infty 2^\nu u(n2^{\nu+1}) < r(n) < n\sum_{\nu=0}^\infty 2^\nu u(n2^\nu).$$

If we put

$$\sum_{\nu=0}^\infty 2^\nu u(n2^\nu) = R_n = u(n) + 2u(2n) + \cdots,$$

then

$$(n/2)\{R_n - u(n)\} = nR_{2n} < r_n < nR_n,$$

and

$$R_n - u_n = 2R_{2n}.$$

It follows that R_n decreases as n increases, and

$$nR_n \to 0.$$

It is now suggestive that for series satisfying the condition (2.7) any useful corrective term must be of the form $c(n+\beta)u_n$, where c and β are constants. We shall discuss a more general problem in the following section; it will give more information on the form of a useful corrective term.

3.1. A generalization of the form $s_n' = s_n + cnu_n$, c a positive constant, is the form $t_n = s_n + \rho_n u_n$, where $\rho_n > 0$. We assume only that $u_n > 0$, and $\sum u_n$ convergent. First of all for what functions ρ_n will t_n yield a closer approximation to s than s_n?
We must have

$$|t_n - s| < s - s_n,$$

hence

$$\rho_n u_n < 2(s - s_n) = 2r_n,$$

that is

$$0 < \rho_n(r_{n-1} - r_n) < 2r_n,$$

or

$$\rho_n r_{n-1} < (2 + \rho_n)r_n.$$

It follows that

$$r_0 \prod_1^n \rho_v < r_n \prod_1^n (2 + \rho_v),$$

or

$$r_n > r_0 \prod_1^n \frac{\rho_v}{2 + \rho_v}.$$

But $r_n \to 0$, hence we must have

$$P_n \equiv \prod_1^n \frac{\rho_v}{2 + \rho_v} \to 0,$$

or

$$\sum_1^\infty \frac{1}{2 + \rho_v} = \infty.$$

In view of $\rho_n > 0$ this condition is equivalent to

$$\sum_1^\infty (1/\rho_v) = \infty. \tag{3.1}$$

Observe that now

$$t_n = s_n + \rho_n u_n, \qquad n = 0, 1, 2, \cdots, \tag{3.2}$$

is the inverse of a regular transform by typical means; according to case 2. of §2 put

$$\frac{\lambda_{n-1}}{p_n} = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} = \rho_n,$$

or

$$\lambda_{n-1}(1 + \rho_n) = \lambda_n \rho_n,$$

thus

$$\lambda_n = \lambda_0 \prod_1^n \frac{1+\rho_\nu}{\rho_\nu},$$

then (see (1.4))

$$s_n = (1/\lambda_n)\textstyle\sum_0^n p_\nu t_\nu$$

and $\lambda_n \to \infty$ by (3.1).

3.2. We now assume that a given sequence ρ_n satisfies the assumptions

$$\rho_n > 0, \text{ and } \textstyle\sum_1^\infty \rho_\nu^{-1} = \infty. \tag{3.3}$$

We wish to determine a corresponding class of convergent series $\sum u_n = s$, $u_n > 0$, for which the transform (3.2) yields a sharper approximation to s than s_n, that is

$$t_n - s = o(s - s_n) \text{ as } n \to \infty.$$

Actually we shall determine the class of series for which

$$\rho_n u_n \sim r_n, \; r_n = \textstyle\sum_{n+1}^\infty u_\nu,$$

that is

$$\rho_n u_n / r_n \to 1, \text{ as } n \to \infty.$$

Accordingly, let

$$\rho_n(r_{n-1} - r_n) = (1 + \gamma_n) r_n, \text{ where } \gamma_n \to 0,$$

or

$$\rho_n r_{n-1} = (1 + \rho_n + \gamma_n) r_n.$$

It follows that

$$r_n = r_0 \prod_1^n \frac{\rho_\nu}{1 + \rho_\nu + \gamma_\nu}, \qquad n \geq 1, \qquad \gamma_n > -1, \tag{3.4}$$

and

$$\begin{aligned} u_n = r_{n-1} - r_n &= r_0 \left(\prod_1^{n-1} \frac{\rho_\nu}{1 + \rho_\nu + \gamma_\nu} \right) \left(1 - \frac{\rho_n}{1 + \rho_n + \gamma_n} \right) \\ &= r_0 \left(\prod_1^n \frac{\rho_\nu}{1 + \rho_\nu + \gamma_\nu} \right) \frac{1 + \gamma_n}{\rho_n}. \end{aligned} \tag{3.5}$$

It is seen from (3.4) that $r_n \downarrow$, and from (3.3) now $r_n \downarrow 0$. Hence $\sum u_n = \sum (r_{n-1} - r_n)$ is convergent. Finally in view of (3.5) $\sum u_n$ converges, and $\rho_n u_n \sim \gamma_n$. For this series now

$$s - t_n = s - s_n - \rho_n u_n = -\gamma_n r_n = o(r_n).$$

Furthermore

$$\operatorname{sgn}(s - t_n) = -\operatorname{sgn}\gamma_n$$

thus if $\gamma_n \downarrow 0$, then $t_n \downarrow s$, and if $\gamma_n \uparrow 0$, then $t_n \uparrow s$. Observe that

$$(3.6)\quad \frac{u_n}{u_{n-1}} = \frac{1+\gamma_n}{1+\gamma_{n-1}}\frac{\rho_{n-1}}{\rho_n}\frac{\rho_n}{1+\rho_n+\gamma_n} = \frac{1+\gamma_n}{1+\gamma_{n-1}}\frac{\rho_{n-1}}{1+\rho_n+\gamma_n} \sim \frac{\rho_{n-1}}{1+\rho_n}.$$

We have now proved the following theorem:

Theorem 3. *Let* $\rho_n > 0$, $\sum_1^\infty \rho_n^{-1} = \infty$, $\gamma_n \to 0$, c *a positive constant, and*

$$(3.7)\quad u_n = c\frac{1+\gamma_n}{\rho_n}\prod_1^n \frac{\rho_v}{1+\rho_v+\gamma_v} > 0;$$

Then $\sum u_n$ *converges, and the transform* $t_n = s_n + \rho_n u_n$ *converges more rapidly to* s, *than* s_n. *Furthermore if* $\gamma_n \downarrow 0$, *then* $t_n \downarrow s$, *and if* $\gamma_n \uparrow 0$, *then* $t_n \uparrow s$.

Example. Let

$$\rho_n = (n+\alpha)/\beta,\ n \geq 1,\ \beta > 0,\ \alpha > 0, \text{ and}$$

$$u_n = \frac{c\beta(1+\gamma_n)}{n+\alpha}\prod_1^n \frac{v+\alpha}{\beta[1+\{(v+\alpha)/\beta\}+\gamma_v]}$$

$$= \frac{c\beta(1+\gamma_n)}{n+\alpha}\prod_1^n \frac{v+\alpha}{\beta+v+\alpha+\beta\gamma_v};$$

so that in particular, for $\gamma_n = 0$

$$u_n = \frac{c\beta}{n+\alpha}\prod_1^n \frac{v+\alpha}{v+\alpha+\beta} \sim \frac{c\beta}{n}\prod_1^n\left(1-\frac{\beta}{v+\alpha+\beta}\right) \sim \frac{c\beta}{n}n^{-\beta} = c\beta n^{-\beta-1}.$$

Note: Similar results hold for integrals of the form $\int_0^\infty u(t)\,dt$.

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(Received November 26, 1948)

AERODYNAMIC THEORY OF OSCILLATING SWEPTBACK WINGS[1]

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1. Introduction. An approximate aerodynamic theory of oscillating wings of finite span without sweepback or with very moderate sweepback has been developed by E. Reissner[2, 3] for incompressible flow. The aspect ratio limitations appear to be no more severe than in applications of lifting line theory to problems of steady flow. The developments presented in this paper comprise the initial phase of a study which has been undertaken to formulate an aerodynamic theory of oscillating wings with *appreciable* sweepback. We consider first a wing of constant chord length with sheared planform as indicated in Fig. 1.

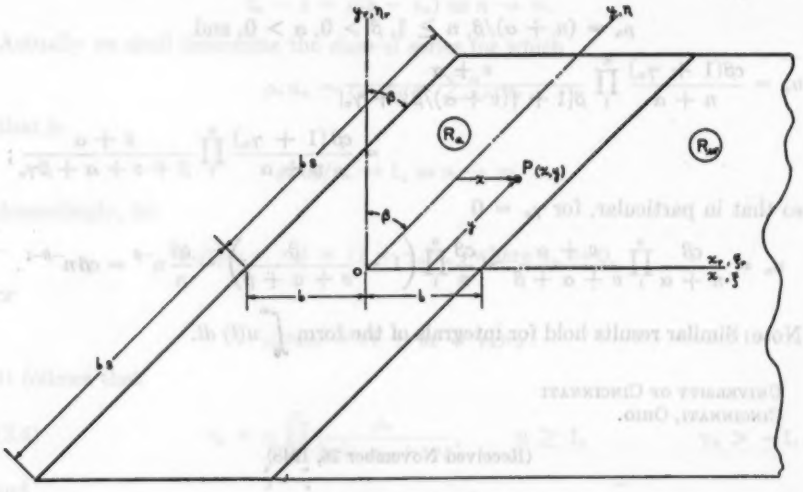

Fig. 1. Sheared Wing

Although such forms appear to be of no direct practical interest, it is anticipated that some features of the subsequent theoretical development will be applicable in studies of the sweptback, V-form wing. Furthermore the theory of the sheared wing may have direct application in the flutter analysis of swept wings of moderately high aspect ratio, since the oscillatory aerodynamic forces

[1] This paper is based on work done under the auspices of the Bureau of Aeronautics, Navy Department, Washington, D. C.

[2] Eric Reissner, *On the general theory of thin airfoils for nonuniform motion*, N.A.C.A. Tech. Note 946 (1944).

[3] Eric Reissner, *Effect of finite span on the airload distributions for oscillating wings; I—aerodynamic theory of oscillating wings of finite span*, N.A.C.A. Tech. Note 1194 (1947).

on one half of the wing will not be influenced appreciably by the state of motion of the other half. Wing displacements and aerodynamic forces are generally small near mid-span; hence it is not essential to obtain a precise determination of the forces in that region for purposes of flutter analysis.

The boundary value problem of the oscillating sheared wing is formulated in oblique coordinates. By introducing certain approximations, depending upon the assumption of sufficiently large aspect ratio, the resulting two-dimensional integral equation is transformed into a one-dimensional equation for the circulation function. As in Reissner's work,[3] the effect of finite span is to introduce an additive correction to the basic function $C(k)$.

The fundamental integral equation of lifting surface theory for the sweptback wing is derived in the final section of the present paper, thus providing a suitable basis for further investigations.

2. Integral equation of lifting surface theory for the sheared wing. Oblique coordinates, (x, y), in the plane of the lifting surface are related to rectangular coordinates, (x_r, y_r), by the relations

$$x_r = x + y \sin \beta, \qquad y_r = y \cos \beta. \tag{1}$$

The flow is rectilinear at $x = -\infty$ with velocity, U, parallel to the x-axis. The density, ρ, is constant, and the pressure in the undisturbed stream is denoted by p_0. Velocity components at an arbitrary point of the half-space $z \geq 0$, where the pressure is $p_0 + p$, are denoted by $U + u, v, w$, referred to rectangular axes. The z-axis is directed upward, perpendicular to the x_r, y_r-plane. Incremental velocities u, v, w are derived from a potential, ϕ, as follows

$$u = \partial\phi/\partial x_r = \partial\phi/\partial x, \; v = \partial\phi/\partial y_r, \; w = \partial\phi/\partial z,$$

and the incremental pressure, p, is related to the velocity potential by the equation

$$-p/\rho = \partial\phi/\partial t + U\partial\phi/\partial x. \tag{2}$$

Airfoil and wake regions are defined as follows

$$R_a\colon\ -b \leqq x \leqq b,\ -sb \leqq y \leqq sb,$$

$$R_w\colon\ b < x,\ -sb \leqq y \leqq sb.$$

Vertical displacements of the lifting surface, positive in the direction of the z-axis, are given by

$$h = \bar{h}(x, y)e^{i\omega t}.$$

Boundary conditions on the flow in the upper half-space are

$$\begin{aligned} w &= \bar{w}_a(x, y)e^{i\omega t} \text{ inside } R_a, \\ p &= 0 \text{ outside } R_a, \end{aligned} \tag{3}$$

where

$$\bar{w}_a = U\partial\bar{h}/\partial x + i\omega\bar{h}.$$

Subsequent derivations are based upon the equation[4]

$$\frac{\partial\phi}{\partial z} = -\frac{1}{4\pi}\iint_{R_a+R_w}\gamma(\xi_r,\eta_r)\left\{\frac{(x_r-\xi_r)}{[(x_r-\xi_r)^2+(y_r-\eta_r)^2+z^2]^{3/2}}\right.$$
$$+\frac{\partial}{\partial y_r}\left[\frac{(y_r-\eta_r)(x_r-\xi_r)}{[(y_r-\eta_r)^2+z^2][(x_r-\xi_r)^2+(y_r-\eta_r)^2+z^2]^{1/2}}\right. \tag{4}$$
$$\left.\left.+\frac{y_r-\eta_r}{(y_r-\eta_r)^2+z^2}\right]\right\}d\xi_r\,d\eta_r,$$

where

$$\gamma(\xi_r,\eta_r) = 2[\partial\phi(\xi_r,\eta_r,\zeta)/\partial\xi_r]_{\zeta=0+}.$$

In Eq. (4) and throughout the remainder of this paper we are concerned only with spatial variations of physical quantities; the bar, which has been used heretofore to distinguish functions of spatial coordinates from time dependent functions, is omitted.

In transforming Eq. (4) to oblique coordinates $\partial/\partial y_r$ is replaced by $-\partial/\partial\eta_r$, where

$$\partial/\partial\eta_r = -\tan\beta\partial/\partial\xi + \sec\beta\partial/\partial\eta. \tag{5}$$

The element of area in oblique coordinates is

$$dA = \cos\beta\,d\xi\,d\eta. \tag{6}$$

Transforming Eq. (4) by means of Eqs. (1), (5), (6), we obtain

$$\frac{\partial\phi}{\partial z} = -\frac{1}{4\pi}\iint_{R_a+R_w}\gamma(\xi,\eta)\left\{\frac{(x-\xi)\cos\beta}{r^3}-\frac{\partial\Phi}{\partial\eta}\right\}d\xi\,d\eta, \tag{7}$$

where

$$\Phi = \frac{(y-\eta)\cos\beta}{[(y-\eta)^2\cos^2\beta+z^2]}\left(\frac{(x-\xi)+(y-\eta)\sin\beta}{r}+1\right),$$
$$r = [(x-\xi)^2+2(x-\xi)(y-\eta)\sin\beta+(y-\eta)^2+z^2]^{\frac{1}{2}}.$$

Integrating by parts with respect to η and noting that $\gamma(\xi,\pm sb) = 0$, we obtain from Eq. (7)

$$\frac{\partial\phi}{\partial z} = -\frac{1}{4\pi}\iint_{R_a+R_w}\frac{\partial\gamma}{\partial\eta}\{\Phi+\Psi\}\,d\xi\,d\eta, \tag{8}$$

with

$$\Psi = \frac{(x-\xi)\cos\beta}{[(x-\xi)^2\cos^2\beta+z^2]}\left(\frac{(x-\xi)\sin\beta+(y-n)}{r}\right).$$

[4] Reissner, Eq. (62), p. 22 of paper quoted in footnote 2.

In the limit as $z \to 0$ Eq. (8) becomes

$$w_a(x, y) \cos \beta = -\frac{1}{4\pi} \oint\limits_{R_a + R_w} \oint \frac{\partial \gamma}{\partial \eta} \left\{ \frac{r_0}{(x - \xi)(y - \eta)} + \frac{1}{y - \eta} \right\} d\xi \, d\eta, \tag{9}$$

where

$$r_0 = \{(x - \xi)^2 + 2(x - \xi)(y - \eta) \sin \beta + (y - \eta)^2\}^{1/2},$$

and, by means of the identity

$$\gamma(\xi, y) = \frac{1}{2} \int_{-b}^{+b} \frac{\partial \gamma(\xi, \eta)}{\partial \eta} \frac{|y - \eta|}{(y - \eta)} \, d\eta,$$

Eq. (9) is transformed into

$$\begin{aligned} w_a(x, y) \cos \beta = &-\frac{1}{2\pi} \oint_{-b}^{\infty} \frac{\gamma(\xi, y)}{x - \xi} \, d\xi \\ &- \frac{1}{4\pi} \int_{-b}^{+b} \oint_{-b}^{\infty} \frac{\partial \gamma / \partial \eta}{y - \eta} \left\{ \frac{r_0}{x - \xi} + 1 - \frac{|y - \eta|}{x - \xi} \right\} d\xi \, d\eta. \end{aligned} \tag{10}$$

Eq. (10) is the fundamental integral equation of lifting surface theory for the sheared wing.

3. Derivation of the integral equation for the circulation function. We next separate the right-hand member of Eq. (10) into two parts, I_1 and I_2, representing the contributions from airfoil and wake regions, respectively. Non-dimensional variables are introduced as follows:

$$x^* = x/b, \; y^* = y/b, \qquad \xi^* = \xi/b, \; \eta^* = \eta/b.$$

The result is

$$w_a(x^*, y^*) \cos \beta = I_1 + I_2, \tag{11}$$

where

$$\begin{aligned} I_1 = &-\frac{1}{2\pi} \oint_{-1}^{1} \frac{\gamma(\xi^*, y^*)}{x^* - \xi^*} \, d\xi^* - \frac{1}{4\pi b} \oint_{-s}^{s} \frac{d\Gamma / d\eta^*}{y^* - \eta^*} \, d\eta^* \\ &- \frac{1}{4\pi} \int_{-s}^{+s} \oint_{-1}^{1} \frac{\partial \gamma}{\partial \eta^*} \left\{ \frac{r_0^* - |y^* - \eta^*|}{(x^* - \xi^*)(y^* - \eta^*)} \right\} d\xi^* \, d\eta^* \end{aligned} \tag{12}$$

$$\begin{aligned} I_2 = &-\frac{1}{2\pi} \int_{1}^{\infty} \frac{\gamma(\xi^*, y^*)}{x^* - \xi^*} \, d\xi^* \\ &- \frac{1}{4\pi} \oint_{-s}^{+s} \int_{1}^{\infty} \frac{\partial \gamma / \partial \eta^*}{y^* - \eta^*} \left\{ \frac{r_0^*}{x^* - \xi^*} + 1 - \frac{|y^* - \eta^*|}{x^* - \xi^*} \right\} d\xi^* \, d\eta^*, \end{aligned} \tag{13}$$

and

$$r_0^* = \{(x^* - \xi^*)^2 + 2(x^* - \xi^*)(y^* - \eta^*) \sin \beta + (y^* - \eta^*)^2\}^{1/2},$$

$$\Gamma = \int_{-b}^{+b} \gamma \, d\xi = b \int_{-1}^{1} \gamma \, d\xi^*.$$

The third integral in Eq. (12) has been neglected in Reissner's theory of unswept wings. The justification, for $\beta = 0$, is as follows: (a) $\partial\gamma/\partial\eta^*$ may be considered constant in the immediate neighborhood of the line $\eta^* = y^*$. When $|y^* - \eta^*| \ll |x^* - \xi^*|$

$$K = \frac{\gamma_0^* - |y^* - \eta^*|}{(x^* - \xi^*)(y^* - \eta^*)} \approx \frac{|x^* - \xi^*|}{(x^* - \xi^*)(y^* - \eta^*)},$$

an odd function of $(y^* - \eta^*)$. (14)

(b) In the region $|y^* - \eta^*| \gg |x^* - \xi^*|$, i.e. over most of the wing,

$$K \approx \frac{(x^* - \xi^*)}{2(y^* - \eta^*)\,|y^* - \eta^*|} = 0\left[\frac{1}{(y^* - \eta^*)^2}\right].$$

In the present case, with $\beta \neq 0$, the approximation given by Eq. (14) is valid in the region $|y^* - \eta^*| \ll |x^* - \xi^*|$. Hence the contribution from that region to the integral in question is still small. However, when $|y^* - \eta^*| \gg |x^* - \xi^*|$ it is found that

$$K \approx \frac{\sin\beta}{|y^* - \eta^*|};$$

consequently it is not permissible to neglect the integral entirely. Following a suggestion due to Reissner, we introduce the approximation

$$K \approx A_\beta\,(y^* - \eta^*) \tag{15}$$

where

$$A_\beta(a) = \frac{1}{2}\int_{-1}^{1} \frac{\{\zeta^2 + 2\zeta a \sin\beta + a^2\}^{1/2} - |a|}{\zeta a}\,d\zeta = \frac{1}{2}\int_{-1}^{1} K(\zeta, a)\,d\zeta.$$

An explicit representation for $A_\beta(a)$ can be obtained by elementary methods of integration. It is found that

$$A_\beta(a) \approx -\sin\beta \log|a|, \quad |a| \ll 1,$$

and

$$A_\beta(a) \to \sin\beta/|a| \text{ as } |a| \to \infty.$$

Also it will be observed that A_β vanishes as $\beta \to 0$. Eq. (12) becomes

$$I_1 = -\frac{1}{2\pi}\int_{-1}^{1} \frac{\gamma(\xi^*, y^*)}{x^* - \xi^*}\,d\xi^* - \frac{1}{4\pi b}\int_{-1}^{1} \frac{d\Gamma}{d\eta^*}\left\{\frac{1}{y^* - \eta^*} + A_\beta(y^* - \eta^*)\right\}d\eta^*. \tag{16}$$

In R_ω, where $\xi > b$, $\xi^* > 1$, we have

$$\gamma(\xi, \eta) = -i\frac{\omega}{U}\,\Gamma(\eta)e^{-i(\omega/U)(\xi - b)}, \qquad \gamma(\xi^*, \eta^*) = -\frac{ik}{b}\,\Gamma(\eta^*)e^{-ik(\xi^* - 1)},$$

$$\frac{\partial\gamma}{\partial\eta^*} = -\frac{ik}{b}\frac{d\Gamma}{d\eta^*}\,e^{-ik(\xi^* - 1)}, \tag{17}$$

with

$$k = \omega b/U.$$

Eqs. (17) are introduced into Eq. (13), with the result

$$I_2 = I_3 + I_4, \tag{18}$$

where

$$I_3 = \frac{ike^{ik}}{2\pi b}\,\Gamma(y^*)\int_1^\infty \frac{e^{-ik\xi^*}}{x^* - \xi^*}\,d\xi^*, \tag{19}$$

$$I_4 = \frac{ike^{ik}}{4\pi b}\oint_{-s}^{s}\int_1^\infty \frac{d\Gamma}{d\eta^*}\,e^{-ik\xi^*}\left\{\frac{r_0^* + (x^* - \xi^*) - |y^* - \eta^*|}{(x^* - \xi^*)(y^* - \eta^*)}\right\}d\xi^*\,d\eta^*. \tag{20}$$

In Eq. (20) we separate the ξ^*-range of integration into two parts as follows

$$\oint_{-s}^{s}\int_1^\infty = \oint_{-s}^{s}\int_{x^*}^\infty - \oint_{-s}^{s}\int_{x^*}^1.$$

Then

$$I_4 = I_5 + I_6, \tag{21}$$

where, with the new variable of integration $\lambda = k(\xi^* - x^*)$,

$$I_5 = \frac{ike^{ik(1-x^*)}}{4\pi b}\oint_{-s}^{s}\frac{d\Gamma}{d\eta^*}\,F_\beta[k(y^* - \eta^*)]\,d\eta^*, \tag{22}$$

and

$$F_\beta(X) = \int_0^\infty e^{-\alpha}\left[\frac{1}{X} + \frac{|X|}{X\lambda} - \frac{\{\lambda^2 - 2\lambda X\sin\beta + X^2\}^{1/2}}{X\lambda}\right]d\lambda. \tag{23}$$

I_6 is given by

$$I_6 = \frac{1 - e^{ik(1-x^*)}}{4\pi b}\oint_{-s}^{s}\frac{d\Gamma/d\eta^*}{y^* - \eta^*}\,d\eta^* - \frac{ike^{ik}}{4\pi b}\oint_{-s}^{s}\int_{x^*}^1 e^{-ik\xi^*}\frac{d\Gamma}{d\eta^*}\,K\,d\xi^*\,d\eta^*. \tag{24}$$

The second integral in Eq. (24) is similar to the third integral in Eq. (12); it may be simplified in the same way by means of Eq. (15), with the result

$$I_6 = \frac{1 - e^{ik(1-x^*)}}{4\pi b}\oint_{-s}^{s}\frac{d\Gamma}{d\eta^*}\left[\frac{1}{y^* - \eta^*} + A_\beta(y^* - \eta^*)\right]d\eta^*. \tag{25}$$

By combining Eqs. (11), (16), (18), (19), (21), (22) and (25) we obtain

$$w_a(x^*, y^*)\cos\beta = -\frac{1}{2\pi}\oint_{-1}^{1}\frac{\gamma(\xi^*, y^*)}{x^* - \xi^*}\,d\xi^* + \frac{ike^{ik}}{2\pi b}\,\Gamma(y^*)\int_1^\infty \frac{e^{-ik\xi^*}}{x^* - \xi^*}\,d\xi^*$$
$$- \frac{e^{ik(1-x^*)}}{4\pi b}\oint_{-s}^{s}\frac{d\Gamma}{d\eta^*}\left\{\frac{1}{y^* - \eta^*} + A_\beta(y^* - \eta^*) - ikF_\beta[k(y^* - \eta^*)]\right\}d\eta^*. \tag{26}$$

It is convenient at this point to introduce the new circulation function[6]

$$\Omega = (1/b)\Gamma e^{i\theta}.$$

Then Eq. (26) becomes

$$w_a(x^*, y^*)\cos\beta = -\frac{1}{2\pi}\oint_{-1}^{1}\frac{\gamma(\xi^*, y^*)}{x^* - \xi^*}\,d\xi^* + \frac{ik\Omega(y^*)}{2\pi}\int_{1}^{\infty}\frac{e^{-ik\xi^*}}{x^* - \xi^*}\,d\xi^* - \frac{e^{-ikx^*}}{4\pi}\oint_{-s}^{s}\frac{d\Omega}{d\eta^*}\left\{\frac{1}{y^* - \eta^*} + A_\beta(y^* - \eta^*) - ikF_\beta[k(y^* - \eta^*)]\right\}d\eta^*. \tag{27}$$

By comparison of Eq. (27) with the corresponding equation for the rectangular wing[6] it is found that the principal results from the earlier work may be applied here with only minor modifications. In the derivation of the integral equation for the circulation function we replace $1/(y^* - \eta^*)$ by $1/(y^* - \eta^*) + A_\beta$, F by F_β and w_a by $w_a\cos\beta$, with the result

$$\Omega(y^*) + \mu(k)\oint_{-s}^{s}\frac{d\Omega}{d\eta^*}\left\{\frac{1}{y^* - \eta^*} + A_\beta(y^* - \eta^*) - ikF_\beta[k(y^* - \eta^*)]\right\}d\eta^* = \Omega_\beta^{(2)}(y^*), \tag{28}$$

where

$$\mu(k) = \frac{J_0(k) - iJ_1(k)}{\pi k\{[J_0(k) - Y_1(k)] - i[J_1(k) + Y_0(k)]\}},$$

$$\Omega_\beta^{(2)} = \frac{4\cos\beta\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\,w_a(\xi^*, y^*)\,d\xi^*}{\pi ik[H_1^{(2)}(k) + iH_0^{(2)}(k)]}. \tag{29}$$

A collocation procedure for the numerical solution of Eq. (28) can be formulated along the lines of Reissner and Stevens' method[7] for unswept wings. It will be necessary to tabulate the functions $A_\beta(a)$, $F_\beta(X)$, and to prepare tables of integrals

$$\int_0^\pi F_\beta[ks(\cos\phi - \cos\theta)]\cos n\theta\,d\theta, \qquad \int_0^\pi A_\beta[s(\cos\phi - \cos\theta)]\cos n\theta\,d\theta,$$

for suitable values of β, k, s, n and ϕ.

4. Lift distribution on oscillating sheared wing. The pressure jump at the lifting surface is giving by

$$-\frac{\Delta p_a(x^*, y^*)}{\rho U} = \lim_{z\to 0+}\frac{p(x^*, y^*, -z) - p(x^*, y^*, z)}{\rho U} = ik\int_{-1}^{x^*}\gamma(\xi^*, y^*)\,d\xi^* + \gamma(x^*, y^*).$$

[5] Reissner[3], Eq. (15), p. 8.

[6] Reissner[3], Eq. (44), p. 19. Set $k_m = 0$, $\zeta = \xi^*$, $z = x^*$, $k_0 = k$, and $K[k_0(y^* - \eta^*)] = 1/(y^* - \eta^*) - ik(y^* - \eta^*)F[k|y^* - \eta^*|]/|y^* - \eta^*|$.

[7] E. Reissner and J. E. Stevens, *Effect of finite span on the airload distributions for oscillating wings; II—methods of calculation and examples of application*, N.A.C.A. Tech. Note 1195 (1947).

For the sheared wing

$$\frac{\Delta p_a}{\rho U} = -\frac{2\cos\beta}{\pi}\int_{-1}^{1}\left\{\sqrt{\frac{1-x^*}{1+x^*}}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\frac{1}{x^*-\xi^*} - ik\Lambda_1(x^*,\xi^*)\right\} w_a(\xi^*,y^*)\,d\xi^* - \frac{2\cos\beta}{\pi}\sqrt{\frac{1-x^*}{1+x^*}}\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\,w_a(\xi^*,y^*)\,d\xi^* + \frac{2\cos\beta}{\pi}\left\{C(k)+\left(\frac{\Omega}{\Omega_\beta^{(2)}}-1\right)\left(C(k)+\frac{iJ_1(k)}{J_0(k)-iJ_1(k)}\right)\right\} \cdot\sqrt{\frac{1-x^*}{1+x^*}}\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\,w_a(\xi^*,y^*)\,d\xi^*, \tag{30}$$

where

$$\Lambda_1(x^*,\xi^*) = \frac{1}{2}\ln\frac{1-x^*\xi^*+\sqrt{1-\xi^{*2}}\sqrt{1-x^{*2}}}{1-x^*\xi^*-\sqrt{1-\xi^{*2}}\sqrt{1-x^{*2}}}.$$

In the two-dimensional case w_a is independent of y and $\Omega = \Omega_\beta^{(2)}$. Except for the cosine factor, Eq. (30) is then identical with the known formula for the two-dimensional problem without sweep. Hence the only modification required in existing formulas for lift and moment on a rigid airfoil is to reduce the aerodynamic coefficients by the factor $\cos\beta$. The chord length is measured in the direction of flight. Angular displacements of wing and control surfaces are measured in a vertical plane, parallel to flight direction.

The effect of finite span is to introduce the additive correction, σ_β, to the basic function $C(k) = F(k) + iG(k)$, where

$$\sigma_\beta = \left(\frac{\Omega}{\Omega_\beta^{(2)}}-1\right)\left[C(k)+\frac{iJ_1(k)}{J_0(k)-iJ_1(k)}\right]. \tag{31}$$

5. Tabulation procedure for the function $F_\beta(X)$. The function

$$F_0(X) = \int_0^\infty e^{-\Omega}\left(\frac{1}{X}+\frac{|X|}{\lambda X}-\frac{\{\lambda^2+X^2\}^{1/2}}{\lambda X}\right)d\lambda$$

is encountered in the theory of unswept wings. $F_0(X)$ is an odd function; it may be expressed in the form

$$F_0(X) = (|X|/X)F(|X|),$$

where

$$F(X) = \int_0^\infty e^{-\Omega}\left(\frac{1}{X}+\frac{1}{\lambda}-\frac{\{\lambda^2+X^2\}^{1/2}}{\lambda X}\right)d\lambda.$$

$F(X)$ has been tabulated by Cicala[8] for positive values of the argument.

Instead of calculating $F_\beta(X)$ directly it is preferable to compute the difference

$$F_0(X) - F_\beta(X) = \int_0^\infty e^{-\Omega}\left(\frac{\{\lambda^2-2\lambda X\sin\beta+X^2\}^{1/2}-\{\lambda^2+X^2\}^{1/2}}{\lambda X}\right)d\lambda. \tag{32}$$

[8] P. Cicala, *Comparison of theory with experiment in the phenomenon of wing flutter*, N.A.C.A. Tech. Memo. 887 (1939).

We may write

$$\{\lambda^2 - 2\lambda X \sin\beta + X^2\}^{1/2} = \{\lambda^2 + X^2\}^{1/2}\left\{1 - \frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right\}^{1/2}, \tag{33}$$

and, since

$$\left|\frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right| \leqq |\sin\beta| < 1,$$

it is permissible to introduce the binomial expansion

$$\left\{1 - \frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right\}^{1/2} = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right)^n.$$

For computational purposes the following approximation is introduced

$$\left\{1 - \frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right\}^{1/2} \approx 1 - \frac{1}{2}\left(\frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right) + \sum_{n=2}^{4} c_n \left(\frac{2\lambda X \sin\beta}{\lambda^2 + X^2}\right)^n. \tag{34}$$

The coefficients c_n are determined by the method of least squares, minimizing the integral

$$\int_c^e \left\{\sqrt{1 - Z} - \left[1 - \tfrac{1}{2}Z + \sum_{n=2}^{4} c_n Z^n\right]\right\}^2 dZ,$$

where

$c = \sin\beta_m$,

β_m = maximum value of β for which theory will be employed.

For $\beta_m = 45°$ the error of the approximation in Eq. (34) is not greater than 0.5%. F. B. Hildebrand has pointed out the desirability of retaining the true value of c_1 from the binomial expansion in Eq. (34).

From Eqs. (32), (33) and (34) we obtain

$$\begin{aligned} F_0(X) - F_\beta(X) = {} & \sum_{n=1}^{4} 2^n c_n X^{n-1} \sin^n\beta \int_0^\infty \frac{\lambda^{n-1}\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{n-1/2}} \\ & - i \sum_{n=1}^{4}{}' 2^n c_n X^{n-1} \sin^n\beta \int_0^\infty \frac{\lambda^{n-1}\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{n-1/2}} = R_\beta - iI_\beta. \end{aligned} \tag{35}$$

After some manipulation there is obtained

$$\begin{aligned} R_\beta = {} & 2c_1 \sin\beta \int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{1/2}} + 4c_2 \sin^2\beta \left(\frac{X}{|X|} - X\int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{1/2}}\right) \\ & + 8c_3 \sin^3\beta \left(X^2\int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{3/2}} - X^4\int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{5/2}}\right) \\ & + 16c_4 \sin^4\beta \left(\frac{2}{15}\frac{X}{|X|} - \frac{X^3}{3}\int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{3/2}} + \frac{X^5}{5}\int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{5/2}}\right), \end{aligned} \tag{36}$$

$$I_\beta = 2c_1 \sin\beta \int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{1/2}} + 4c_2 \sin^2\beta \left(X \int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{1/2}} \right)$$
$$+ 8c_3 \sin^3\beta \left(X^2 \int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{3/2}} - X^4 \int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{5/2}} \right) \qquad (37)$$
$$+ 16c_4 \sin^4\beta \left(\frac{X^3}{3} \int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{3/2}} - \frac{X^5}{5} \int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{5/2}} \right).$$

The integrals involving cos λ may be expressed in terms of modified Bessel functions[9] as follows:

$$\int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{1/2}} = K_0(|X|), \qquad \int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{3/2}} = \frac{K_1(|X|)}{|X|},$$
$$\int_0^\infty \frac{\cos\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{5/2}} = \frac{K_2(|X|)}{3X^2} = \frac{K_0(|X|)}{3X^2} + \frac{2K_1(|X|)}{3|X|^3}. \qquad (38)$$

If we introduce the notation

$$T_n(X) = |X|^n \int_0^\infty \frac{\sin\lambda\, d\lambda}{\{\lambda^2 + X^2\}^{n+1/2}}, \qquad n = 0, 1, 2, \cdots, \qquad (39)$$

then it can be easily shown that

$$T_{n+1} = -\frac{1}{(4n^2 - 1)|X|^n} + \frac{T_{n-1}}{(4n^2 - 1)} + \frac{2nT_n}{(2n + 1)|X|}. \qquad (40)$$

T_0 and T_1 have been discussed elsewhere;[10] for $X > 0$

$$T_0(X) = K_0(X) \int_0^x I_0(\xi)\, d\xi - I_0(X) \int_0^x K_0(\xi)\, d\xi + \frac{\pi}{2} I_0(X), \qquad (41)$$

$$T_1(X) = -T_0' = K_1(X) \int_0^x I_0(\xi)\, d\xi + I_1(X) \int_0^x K_0(\xi)\, d\xi - \frac{\pi}{2} I_1(X). \qquad (42)$$

The integrals occurring in Eqs. (41) and (42) have been tabulated.[11]

6. Integral equation of lifting surface theory for the sweptback wing. It appears desirable, in the analysis of the V-form wing, to treat symmetrical and antisymmetrical motions separately. Because of the geometric symmetry of the wing, this makes it possible to formulate the basic integral equation for one half of the wing only, and thereby to simplify the region of integration.

As indicated in Fig. 2, oblique coordinates are to be employed, with the y-axis parallel to the leading edge of the right-hand half of the wing. We return to Eq. (4), which is valid for a wing of any planform. The airfoil region is divided

[9] Gray, Mathews and MacRobert, *A treatise on Bessel functions and their applications to physics*, 2nd ed., Macmillan and Co., Ltd., 1931, Eq. (35), p. 52.

[10] W. P. Jones, *Wind tunnel interference effect on the values of experimentally determined derivative coefficients for oscillating aerofoils*, Brit., A.R.C., R. & M. 1912 (1943), p. 9.

[11] V. R. Bursian and V. A. Fok, Akad. Nauk., Leningrad, Fiziks-Mat. Institut, *Trudy* (*Travaux*) (1)2, 6–10 (1931).

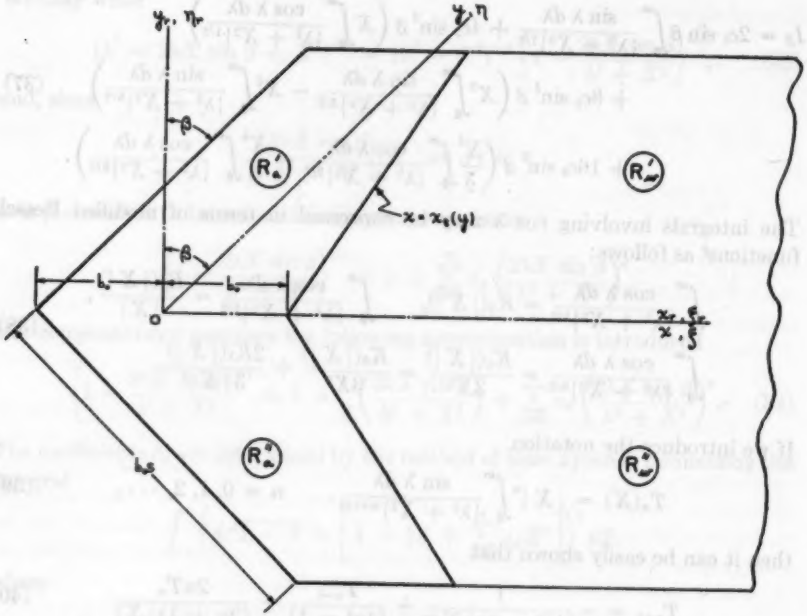

FIG. 2. Sweptback Wing

into a right-hand region, R_s', and a left-hand region, R_s''; R_w is divided similarly into R_w' and R_w''. On account of the symmetry of the airfoil, Eq. (4) may be written

$$\frac{\partial \phi}{\partial z} = -\frac{1}{4\pi} \iint_{R_s'+R_w'} \gamma(\xi_r, \eta_r) \left\{ \frac{x_r - \xi_r}{r'^3} + \frac{\partial \Phi'}{\partial y_r} \right\} d\xi_r \, d\eta_r$$
$$- \frac{1}{4\pi} \iint_{R_s'+R_w'} \gamma(\xi_r, -\eta_r) \left\{ \frac{x_r - \xi_r}{r''^3} + \frac{\partial \Phi''}{\partial y_r} \right\} d\xi_r \, d\eta_r, \tag{43}$$

where

$$r' = \{(x_r - \xi_r)^2 + (y_r - \eta_r)^2 + z^2\}^{1/2},$$
$$r'' = \{(x_r - \xi_r)^2 + (y_r + \eta_r)^2 + z^2\}^{1/2},$$
$$\Phi' = \frac{y_r - \eta_r}{(y_r - \eta_r)^2 + z^2} \left[\frac{x_r - \xi_r}{r'} + 1 \right],$$
$$\Phi'' = \frac{y_r + \eta_r}{(y_r + \eta_r)^2 + z^2} \left[\frac{x_r - \xi_r}{r''} + 1 \right].$$

The relations

$$\gamma(\xi_r, -\eta_r) = \begin{cases} \gamma(\xi_r, \eta_r) \text{ for symmetrical motions,} \\ -\gamma(\xi_r, \eta_r) \text{ for antisymmetrical motions,} \end{cases} \tag{44}$$

are introduced into Eq. (43) and the resulting equation is transformed to oblique coordinates by means of Eqs. (1), (5) and (6). Since

$$\frac{\partial \Phi'}{\partial y_r} = -\frac{\partial \Phi'}{\partial \eta_r} = \tan\beta \frac{\partial \Phi'}{\partial \xi} - \sec\beta \frac{\partial \Phi'}{\partial \eta},$$

$$\frac{\partial \Phi''}{\partial y_r} = \frac{\partial \Phi''}{\partial \eta_r} = -\tan\beta \frac{\partial \Phi''}{\partial \xi} + \sec\beta \frac{\partial \Phi''}{\partial \eta},$$

and, by straightforward calculation,

$$\frac{\partial \Phi'}{\partial \xi} = -\frac{(y-\eta)\cos\beta}{r'^3}, \qquad \frac{\partial \Phi''}{\partial \xi} = -\frac{(y+\eta)\cos\beta}{r''^3},$$

it follows that

$$\begin{aligned}\frac{\partial \phi}{\partial z} = &-\frac{1}{4\pi}\iint_{R'_a+R'_a} \gamma(\xi,\eta)\left\{\frac{(x-\xi)\cos\beta}{r'^3} - \frac{\partial \Phi'}{\partial \eta}\right\} d\xi\, d\eta \\ &\mp \frac{1}{4\pi}\iint_{R'_a+R'_a} \gamma(\xi,\eta)\left\{\frac{(x-\xi)\cos\beta + 2y\sin\beta\cos\beta}{r''^3} + \frac{\partial \Phi''}{\partial \eta}\right\} d\xi\, d\eta.\end{aligned} \tag{45}$$

The upper or lower sign, preceding the second integral in Eq. (45), is to be chosen according as the motion is symmetrical or antisymmetrical. Both integrals in Eq. (45) are transformed by integration by parts with respect to η, and the substitution $\gamma(\xi, b_0 s) = 0$ is made. The limiting form of the resulting equation as $z \to 0$ is

$$\begin{aligned}w_a(x,y)\cos\beta = &-\frac{1}{4\pi}\oint_{-b_0}^{\infty} \gamma(\xi,0)\left\{\frac{\bar{r}'_0}{(x-\xi)y} + \frac{1}{y} - f_1\right\} d\xi \\ &- \frac{1}{4\pi}\oint\oint_{R'_a+R'_a} \frac{\partial \gamma}{\partial \eta}\left\{\frac{r'_0}{(x-\xi)(y-\eta)} + \frac{1}{y-\eta} - f_1\right\} d\xi\, d\eta \\ &\pm \frac{1}{4\pi}\oint_{-b_0}^{\infty} \gamma(\xi,0)\left\{\frac{\bar{r}''_0}{[(x-\xi)+2y\sin\beta]y} + \frac{1}{y} + f_2\right\} d\xi \\ &\pm \frac{1}{4\pi}\int\oint_{R'_a+R'_a} \frac{\partial \gamma}{\partial \eta}\left\{\frac{r''_0}{[(x-\xi)+2y\sin\beta](y+\eta)} + \frac{1}{y+\eta} + f_2\right\} d\xi\, d\eta,\end{aligned} \tag{46}$$

where

$$r'_0 = \{(x-\xi)^2 + 2(x-\xi)(y-\eta)\sin\beta + (y-\eta)^2\}^{1/2},$$

$$\begin{aligned}r''_0 &= \{(x-\xi)^2 + 2(x-\xi)(y-\eta)\sin\beta + (y-\eta)^2\sin^2\beta + (y+\eta)^2\cos^2\beta\}^{1/2} \\ &= \{[(x-\xi)+2y\sin\beta]^2 - 2\sin\beta[(x-\xi)+2y\sin\beta](y+\eta) + (y+\eta)^2\}^{1/2},\end{aligned}$$

$$\begin{aligned}\bar{r}'_0 &= \bar{r}''_0 = \{(x-\xi)^2 + 2(x-\xi)y\sin\beta + y^2\}^{1/2} \\ &= \{[(x-\xi)+2y\sin\beta]^2 - 2\sin\beta[(x-\xi)+2y\sin\beta]y + y^2\}^{1/2}.\end{aligned}$$

f_1 and f_2 are arbitrary functions of $(x - \xi)$, y, subject only to the requirement that the integrals in Eq. (46) shall be convergent. We make the obvious choice

$$f_1 = \frac{\bar{r}_0'}{(x-\xi)y} + \frac{1}{y},$$

$$f_2 = -\frac{\bar{r}_0''}{[(x-\xi)+2y\sin\beta]y} - \frac{1}{y}, \tag{47}$$

and Eq. (46) becomes

$$w_a(x,y)\cos\beta = -\frac{1}{4\pi}\iint_{R_a'+R_w'}\frac{\partial\gamma}{\partial\eta}\left\{\frac{r_0'}{(x-\xi)(y-\eta)} + \frac{1}{y-\eta} - \frac{\bar{r}_0'}{(x-\xi)y} - \frac{1}{y}\right\}d\xi\,d\eta \pm \frac{1}{4\pi}\iint_{R_a'+R_w'}\frac{\partial\gamma}{\partial\eta}\left\{\frac{r_0''}{[(x-\xi)+2y\sin\beta](y+\eta)} + \frac{1}{y+\eta} - \frac{\bar{r}_0''}{[(x-\xi)+2y\sin\beta]y} - \frac{1}{y}\right\}d\xi\,d\eta. \tag{48}$$

Finally we employ the identity

$$\gamma(\xi,y) = \frac{1}{2}\int_0^{b_0 s}\frac{\partial\gamma}{\partial\eta}\left\{\frac{|y-\eta|}{y-\eta} - 1\right\}d\eta,$$

to transform Eq. (48), with the result

$$w_a(x,y)\cos\beta = -\frac{1}{2\pi}\int_{-b_0}^{\infty}\frac{\gamma(\xi,y)}{x-\xi}\,d\xi - \frac{1}{4\pi}\iint_{R_a'+R_w'}\frac{\partial\gamma}{\partial\eta}\left\{\frac{r_0' - |y-\eta|}{(x-\xi)(y-\eta)} + \frac{1}{y-\eta} - \frac{\bar{r}_0' - y}{(x-\xi)y} - \frac{1}{y}\right\}d\xi\,d\eta \pm \frac{1}{4\pi}\iint_{R_a'+R_w'}\frac{\partial\gamma}{\partial\eta}\left\{\frac{r_0''}{[(x-\xi)+2y\sin\beta](y+\eta)} + \frac{1}{y+\eta} - \frac{\bar{r}_0''}{[(x-\xi)+2y\sin\beta]y} - \frac{1}{y}\right\}d\xi\,d\eta, \tag{49}$$

Eq. (49) is the fundamental integral equation of lifting surface theory for the sweptback wing, expressed in oblique coordinates. As given here, it relates the vertical component of fluid velocity, $w_a(x, y)$, in the right-hand half of the airfoil region

$$R_a'\colon -b_0 \leqq x \leqq x_t(y),\ 0 \leqq y \leqq b_0 s,$$

to the distribution of vorticity, $\gamma(\xi, \eta)$, in the region

$$R_a' + R_w'\colon -b_0 \leqq \xi,\ 0 \leqq \eta \leqq b_0 s.$$

The distribution of induced velocity, $w_a(x, y)$, along a particular chordwise line, y = const., is composed of two parts. The contribution of the first integral in Eq. (49) is the velocity that would exist if the flow were two-dimensional, with the vorticity distribution $\gamma(\xi, \eta) = \gamma(\xi, y)$. Three dimensional effects are represented by the second and third integrals.

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(Received March 18, 1949)